SIMULATION OF TRAFFIC DYNAMICS

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1 INTRODUCTION

To control and influence traffic flows on highways detector equipments and variable message sign systems have been installed in the recent years. Traffic control systems are based on the idea to avoid traffic instabilities and to homogenize the traffic flow in such a way that the risk of accidents is minimized and the mean velocity or the traffic flow is maximized. Typical control measures are speed limitations, no-passing zones, keep-in-lane recommendations, on-ramp-regulations or alternative route recommendations. The traffic control systems need the evaluation of the measured traffic data, a short time prediction of the traffic situation and traffic simulations for possible control measures without significant time delay. With the increasing need for optimized traffic measures, the development of fast and robust numerical methods for traffic simulations with adaption to measured traffic data becomes more and more important.

Traffic measurements are usually carried out at fixed intersections of a highway using installed detector equipments. The traffic flow, the mean velocity and the traffic density are calculated directly from the detector data. It is not ensured that the calculated traffic data particularly for jam situations and ramp flows satisfy the condition of conservation of the number of vehicles in the traffic stream. This is, however, a basic assumption for numerical traffic simulations with adaptions to measured traffic data.

A wide range of different mathematical models of traffic flow has been developed by scientist from mathematics, physics and engineering. There are two main directions of development: microscopic and macroscopic traffic modelling. In microscopic modelling, the motion of each vehicle of a traffic stream is considered depending on the motion of the preceding vehicle and the individual driving behaviour can be taken into account. In macroscopic modelling, the motion of a traffic stream is described in analogy to fluid dynamics by partial differential equations which have to be solved numerically. The basic microscopic and macroscopic traffic models are single-lane models which are able to describe the spectrum of observed phenomena in real traffic on highways. The key problem in practical applications is the identification of model parameters which depend significantly on the traffic scenario.

2 TRAFFIC MEASUREMENTS

A traffic stream of a highway consists of partial traffic streams on the lanes of the highway. Using detector equipments at fixed intersections of the highway, the traffic flow and the mean velocity are measured within a certain time interval. These local measurement data have to be evaluated and prepared carefully with the aim to generate suitable data for traffic simulations. Simple methods of the evaluation of local measurement data including their problems are described below.

Trajectories of vehicles : A single lane of a highway with moving vehicles is considered. The position of vehicle i at time t is denoted by $x_i(t)$. The function $x_i(t)$ is called the trajectory of vehicle i. The time derivative dx_i/dt is the velocity of vehicle i. The velocity is always non negative so that a trajectory is a monotonously increasing function. Trajectories of different vehicles do not intersect each other. It is useful to visualize trajectories in a space-time diagram.

Trajectories of vehicles on a single lane can be computed from aerial photographs. Figure 2.1 shows the trajectories of a moving traffic jam in a space-time diagram. The jam region can be recognized by the region of trajectories which are nearly horizontal and close together. The vehicles in the jam region are moving very slowly with short distances to their predecessors. The jam wave moves backward.

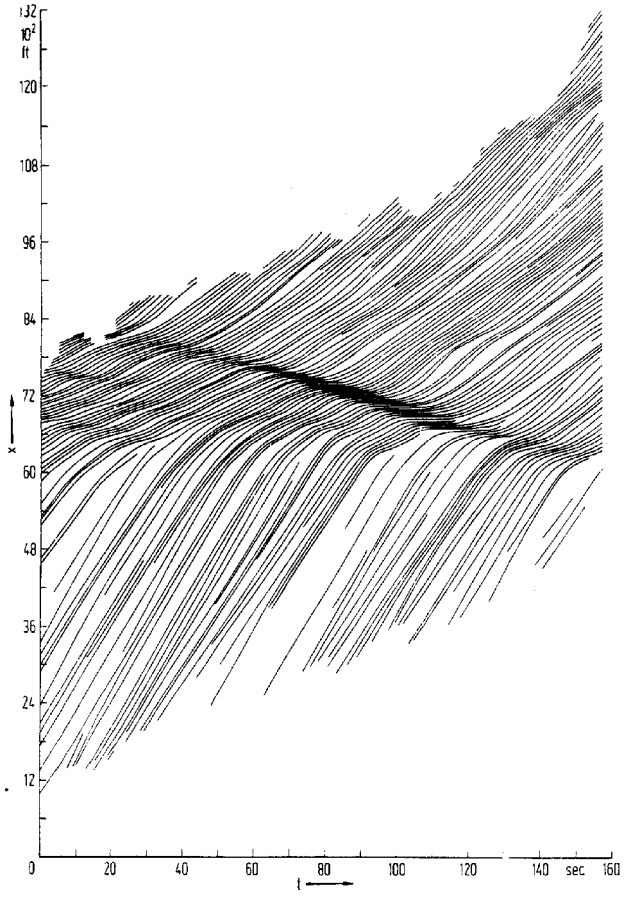


Figure 2.1 : Trajectories of vehicles from aerial photographs [1]

Local traffic measurements : Local traffic measurements are usually carried out at fixed intersections of a highway using installed detectors for each lane. A detector of lane j counts the vehicles which pass the intersection position x within a prescribed time interval from t to $t + \Delta T$. Additionally the velocities of the detected vehicles are measured and the mean velocity is determined.

$\Delta N_j\,(x, t)$	number of vehicles on lane j during ΔT
$v_j\,(x, t)$	mean velocity of vehicles on lane j during ΔT

The evaluation of local measurement data depends significantly on the time interval ΔT. If ΔT is too small, random fluctuations are dominant. If ΔT is too large, stop and go waves are smoothed away. In the literature it is proposed that ΔT should be chosen in the range from 0.5 to 10 minutes. $\Delta T = 1$ min is a suitable choice for using local measurement data in traffic simulations.

Traffic flow and density : The traffic flow and the traffic density on a single lane are computed from the local measurement data. The traffic flow is defined as the number of vehicles per time unit and is calculated as follows :

$$q_j\,(x, t) = \frac{\Delta N_j\,(x, t)}{\Delta T} \tag{2.1}$$

The mean traffic density is defined as the number of vehicles per length unit and can be calculated approximately from the traffic flow and the mean velocity :

$$\varrho_j\,(x, t) = \frac{\Delta N_j\,(x, t)}{\Delta x_j} \approx \frac{\Delta N_j\,(x, t)}{\Delta T \cdot v_j\,(x, t)} = \frac{q_j\,(x, t)}{v_j\,(x, t)} \tag{2.2}$$

The approach $\Delta x_j \approx \Delta T \cdot v_j\,(x, t)$ is only valid, if all detected vehicles are moving with approximately the same velocity during the time interval ΔT. This requirement is not satisfied for stop and go traffic. Hence, the mean densities calculated from local measurement data are not realistic in this case.

Macroscopic traffic data : The macroscopic traffic data at each intersection of a highway are computed from the corresponding traffic data of all lanes. The total traffic flow is the sum of the traffic flows of all lanes :

$$q\,(x, t) = \sum_j q_j\,(x, t) \tag{2.3}$$

Two different methods can be used for the calculation of the total mean velocity. In the first method the total mean velocity is calculated as the weighted arithmetic average of the mean velocities of all lanes. In the second method the weighted harmonic average is used instead of the weighted arithmetic average in the first method.

$$v_I\,(x, t) = \sum_j v_j\,(x, t)\ \frac{q_j\,(x, t)}{q\,(x, t)} \tag{2.4}$$

$$\frac{1}{v_{II}(x,t)} = \sum_j \frac{1}{v_j(x,t)} \frac{q_j(x,t)}{q(x,t)} \tag{2.5}$$

The total mean density is calculated from the total traffic flow and the total mean velocity as follows :

$$\varrho_I(x,t) = \frac{q(x,t)}{v_I(x,t)} = \frac{q^2(x,t)}{\sum_j v_j(x,t) \cdot q_j(x,t)} \tag{2.6}$$

$$\varrho_{II}(x,t) = \frac{q(x,t)}{v_{II}(x,t)} = \sum_j \frac{q_j(x,t)}{v_j(x,t)} = \sum_j \varrho_j(x,t) \tag{2.7}$$

If the mean velocities of all lanes are equal, both methods lead to the same results for the total mean velocity and density. Otherwise the first method always leads to a larger total mean velocity and to a lower total mean density than the second method.

$$\begin{aligned} v_I(x,t) &\geq v_{II}(x,t) \\ \varrho_I(x,t) &\leq \varrho_{II}(x,t) \end{aligned} \tag{2.8}$$

Both methods are widely used in traffic engineering. Their results may be significantly different. The calculation of the total mean velocity is consistent with the statistical rule of computing mean values in the first method but not in the second method. The calculation of the total mean density is consistent with the definition of traffic densities in the second method but not in the first method.

Problem : The computation of macroscopic traffic data from local measurement data using the simple methods described above is problematic. The basic problems are :

- The computed traffic densities are not realistic if the traffic flows and velocities are low.
- It is not ensured, that the computed macroscopic traffic data satisfy the condition of mass conservation with a sufficient accuracy.

Figure 2.2 shows time dependent functions of typical macroscopic traffic data which are computed from local measurements data at an intersection of a highway.

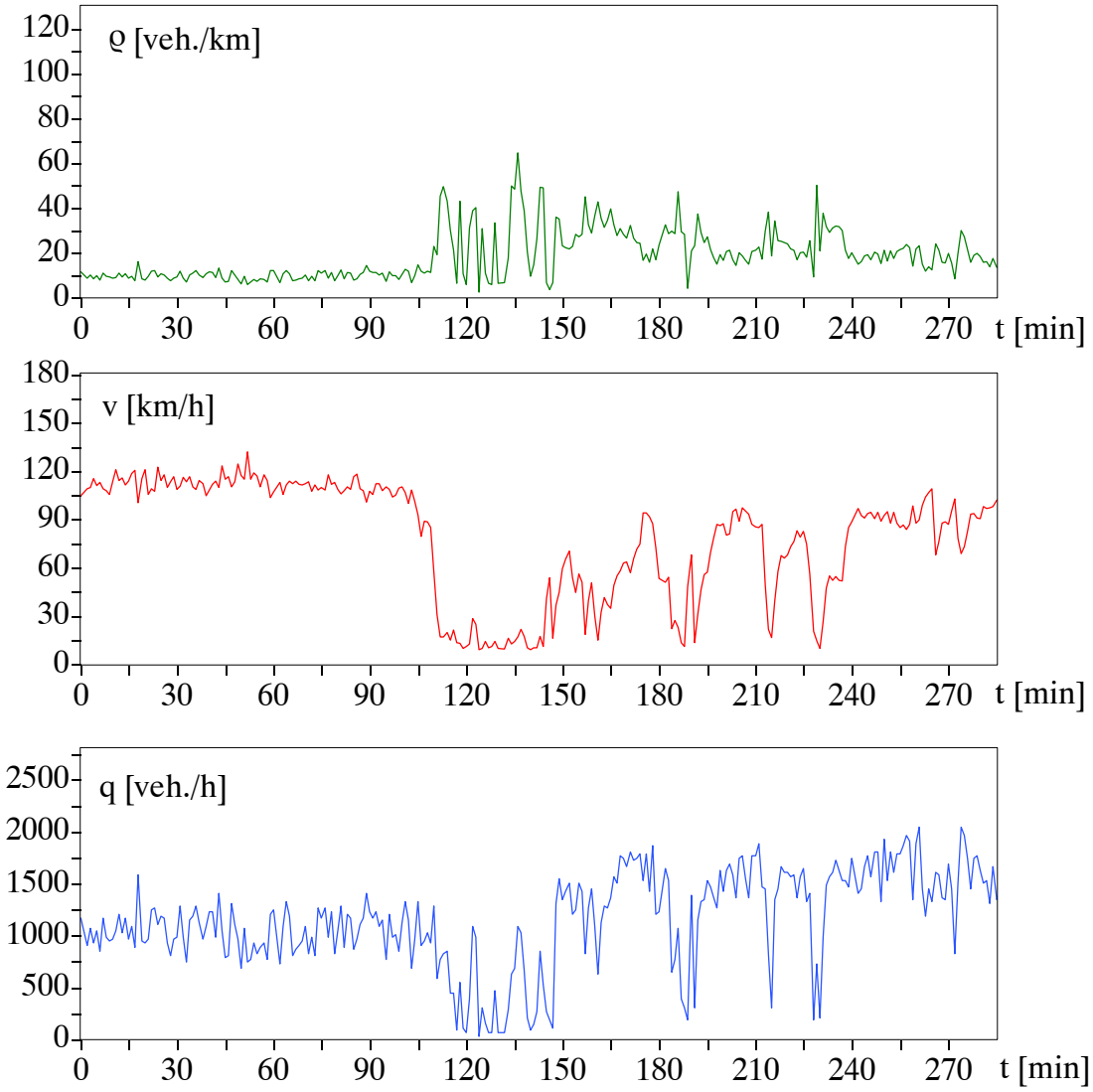


Figure 2.2: Traffic density, velocity, flow computed from local measurement data

3 MICROSCOPIC TRAFFIC MODELLING

In microscopic traffic modelling, the motion of each vehicle in a traffic stream is considered. The driving behaviour of a vehicle depends significantly on the motion of the preceding vehicle and can be specified by simplified rules or interaction relationships. A basic model of cellular automata and a basic dynamic model of congested traffic are described below.

3.1 Cellular Automaton Model

The cellular automaton model for microsimulations of traffic flow was developed by Nagel and Schreckenberg [2]. The basic model is a single-lane model which is discrete in space and time. The driving behaviour is specified by only a few simple rules without losing the essential phenomena in traffic dynamics. This single-lane model can be extended to more complex models, for example to multi-lane models with on-and off-ramps and different types of vehicles.

Traffic state : A single lane with moving vehicles is considered. The lane is subdivided into cells of constant length Δx. At time t each cell is either empty or occupied by one vehicle. Each vehicle moves with a certain velocity which is specified by an integer number υ in the range from 0 to υ_{max}. This integer velocity is defined as the number of cells which are passed by the vehicle during a prescribed time interval Δt. Each vehicle has a certain distance to the preceding vehicle which is specified by an integer number $s \geq 1$. This integer distance is defined as the number of empty cells between the vehicles increased by 1. The velocity υ of a vehicle at time t must always be less than the distance s to its predecessor, so that safe driving is ensured.

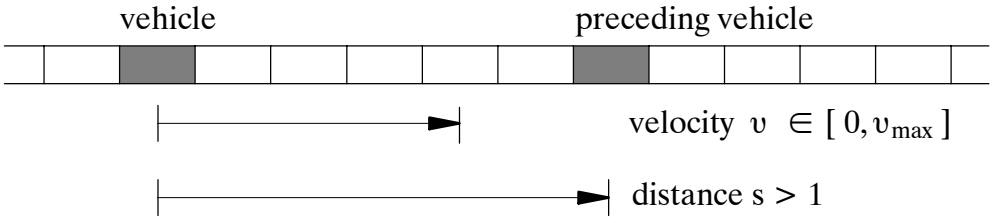


Figure 3.1 : Traffic state on a single lane

Rules of traffic dynamics : The traffic state at time $t + \Delta t$ is computed from the traffic state at time t using the following four rules consecutively :

(1) Acceleration : if $(s > \upsilon + 1)$ then $\upsilon := \text{Min}\{\upsilon + 1, \upsilon_{max}\}$
If the distance s of a vehicle to its predecessor is larger than $\upsilon + 1$ and the velocity υ is lower than υ_{max} , then the velocity is increased by 1.

(2) Braking: if $(s < \upsilon + 1)$ then $\upsilon := s - 1$
If the distance s of a vehicle to its predecessor is shorter than $\upsilon + 1$, then the velocity υ is reduced to the safe distance $s - 1$.

(3) Randomization: with probability p do $\upsilon := \text{Max}\{0, \upsilon - 1\}$
If the velocity υ of a vehicle is greater than 0, the velocity υ is decreased by 1 with a prescribed probability p.

(4) Motion: Each vehicle moves υ cells forward.

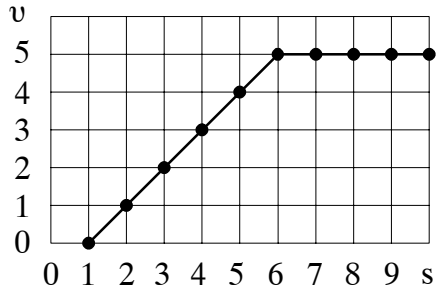
Space and time scaling : Traffic simulations require a suitable space and time scaling of the cellular automaton model. The cell length Δx should be equal to the average length of a vehicle. The maximum velocity υ_{max} should represent the average velocity v_f in free traffic. The time interval Δt depends on $\Delta x, \upsilon_{max}$ and v_f. Typical values for space and time scaling are :

$$\Delta x = 7.5 \text{ m} \qquad v_f = 130 \text{ km/h} \qquad \upsilon_{max} = 5$$

$$\Delta t = \Delta x \, \upsilon_{max} / v_f = 1.04 \text{ s}$$

Initial and boundary conditions : Theoretical investigations of traffic models are often carried out for closed lanes (circuit) without boundaries. In this case only an initial state of the traffic simulations has to be specified. Practical computations, however, are performed for open lanes with an upstream and downstream boundary. In addition to the initial state, boundary conditions have to be specified. Boundary conditions can be formulated and implemented for different traffic situations.

Deterministic model : The cellular automaton model without randomized velocity reduction (rule 3) is a deterministic model. Starting with an arbitrary initial state for a closed lane the traffic simulation reaches a steady state after a finite number of time steps. Steady states for free and congested traffic are quite different. In free traffic all vehicles are moving with maximum velocity υ_{max}. In congested traffic each vehicle is moving with the safety distance $s - 1$ to its predecessor. In steady states the velocity is a function of the distance which is shown in Figure 3.



Free traffic:
$\upsilon = \upsilon_{max} \qquad s \geq \upsilon_{max} + 1$

Congested traffic:
$\upsilon = s - 1 \qquad 1 \leq s \leq \upsilon_{max}$

Figure 3.2 : Velocity-distance function of steady states

Stochastic model : The cellular automaton model with randomized velocity reduction (rule 3) is a stochastic model. The essential parameter of this model is the probability p in rule 3. In connection with rule 1 and 2 the randomization rule 3 reflects three different pattern of driving behaviour: fluctuations at maximum velocity in free traffic, retarded accelerations in stop-and-go traffic and over-reactions at braking.

The velocity-distance relationship of the stochastic model can be described by a discrete probability distribution depending on the parameter p. Figure 3.3 shows the mean velocity for each discrete distance which is computed from a traffic simulation on a closed lane with 100 cells, 16 vehicles, 5000 time steps and p = 50%.

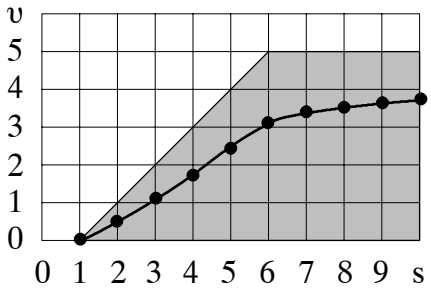


Figure 3.3 : Velocity-distance relationship of a stochastic traffic simulation with p = 50%

3.2 Dynamic Model of Traffic Congestions

The kinematics of vehicles form the basis of microscopic traffic modelling. There are two different approaches of vehicle motions in traffic streams on a single lane. The first approach is that each vehicle must maintain the safe distance to the preceding vehicle which depends on the relative velocities of the successive vehicles. The models based on this approach are called follow-the-leader models. The second approach is that each vehicle has a desired velocity which depends on the distance to the preceding vehicle. A simple model based on this approach was developed by Bando et al. [3] and is described below.

Traffic state : A single lane with moving vehicles i = 1, ..., n is considered. The position of a vehicle i at time t is denoted by x_i and the velocity by $v_i \geq 0$. The distance from a vehicle i to the preceding vehicle i+1 is $s_i = x_{i+1} - x_i$.

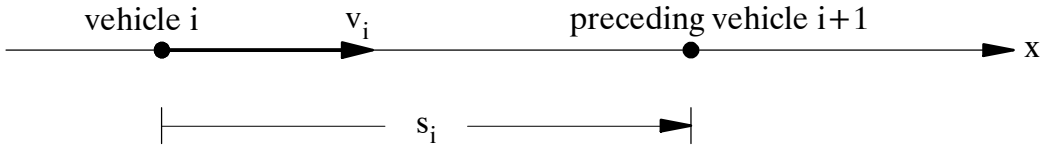


Figure 3.4 : Traffic state on a single lane

Dynamic equations : It is assumed, that each vehicle has the same desired velocity V (s) which depends on the distance s. The acceleration of vehicle i is controlled by the driver in such a way, that the current velocity v_i is adapted to the desired velocity $V(s_i)$ within a certain time τ. From these assumptions the following dynamic equations for each vehicle i are obtained :

$$\frac{dx_i}{dt} = v_i \qquad v_i \geq 0 \qquad i = 1, ..., n \tag{3.1}$$

$$\frac{dv_i}{dt} = \frac{V(s_i) - v_i}{\tau} \qquad s_i = x_{i+1} - x_i \qquad i = 1, ..., n \tag{3.2}$$

The adaption time τ is usually chosen as a constant value in the range from 0.5 to 2.0 sec. It seems to be more realistic to specify the adaption time τ in dependence of $V(s_i) - v_i$, so that adaptions to lower velocities by braking and adaptions to higher velocities by accelerating can be modelled differently (see also rule 1 and 2 of the cellular automaton model).

Velocity-distance function : When the distance of a vehicle to its predecessor becomes smaller the vehicle has to reduce its velocity by braking. When the distance becomes larger the vehicle can accelerate without exceeding a maximum velocity V_{max}. Hence, the velocity distance function V (s) is a monotonously increasing function with V (0) = 0 and $V(s) \rightarrow V_{max}$ for $s \rightarrow \infty$. In [3] the following velocity-distance function is proposed:

$$V(s) = V_{max} \frac{\tanh (d - 2) + \tanh 2}{1 + \tanh 2} \qquad (3.3)$$

$d = s / D$ normalized distance

Numerical simulations : The dynamic equations are solved numerically. Using the explicit Euler method, the traffic state a time $t + \Delta t$ is computed from the traffic state at time t with $\Delta t < \tau$ as follows:

$$x_i (t + \Delta t) = x_i (t) + \Delta t \cdot v_i (t)$$
$$v_i (t + \Delta t) = v_i (t) \left(1 - \frac{\Delta t}{\tau} \right) + \frac{\Delta t}{\tau} V (s_i (t)) \qquad (3.4)$$

Equilibrium state : A traffic state is called an equilibrium state, if all vehicles move with the same distance s to their predecessors and the same velocity v = V (s). The dynamic equations are satisfied for each equilibrium state. An equilibrium state is stable, if small deviations from the equilibrium distances decrease with time evolution. Otherwise the equilibrium state is unstable. The stability condition can be determined using the methods of linear stability theory. It depends on the derivative $V'(s)$ of the velocity-distance function and the adaption time τ:

$$\text{stability:} \qquad V'(s) < 1 / 2\tau \qquad (3.5)$$

If the stability condition for an equilibrium state is not satisfied, small perturbations are producing moving jams.

4 MACROSCOPIC TRAFFIC MODELLING

Macroscopic traffic modelling is based on the assumption that a traffic stream on a single lane can be considered as a continuum of moving particles. The macroscopic traffic states are described by continuous functions of the traffic density and the traffic velocity in the space-time domain. They can be transformed into equivalent microscopic traffic states so that the results of macroscopic traffic models can be evaluated on a microscopic level. The governing equations of macroscopic traffic dynamics are the continuity equation for the density and the equation of motion for the velocity. The main problem is the formulation of realistic driving forces in the equation of motion. It is assumed in all various formulations that the internal driving force describes the adaption of the current velocity to a prescribed equilibrium velocity with a certain adaption time. This approach of the internal driving force is equivalent to the approach in microscopic modelling (see section 3.2). The partial differential equations, the equilibrium state as well as the initial and boundary conditions of macroscopic traffic dynamics are described in the following sections.

4.1 Macroscopic and Microscopic Traffic States

Macroscopic traffic state : A macroscopic traffic state on a single lane at time t is specified by the continuous density function $\varrho\,(x,t) \geq 0$ and the continuous velocity function $v\,(x,t) \geq 0$. The density function $\varrho\,(x,t)$ has an upper limit ϱ_{max} which represents the maximum number of vehicles per length unit. The number of vehicles $N\,(a,b,t)$ in the spatial interval $[a,b[$ at time t and the corresponding average velocity $\bar{v}\,(a,b,t)$ of the vehicles are calculated as follows:

$$N\,(a,b,t) = \int_a^b \varrho\,(x,t)\,dx \qquad 0 \leq \varrho\,(x,t) \leq \varrho_{max} \tag{4.1}$$

$$\bar{v}\,(a,b,t) = \int_a^b v\,(x,t)\,dx \,/\, (b-a) \qquad v\,(x,t) \geq 0 \tag{4.2}$$

Microscopic traffic state : A macroscopic traffic state can be transformed into an equivalent microscopic traffic state using the following procedure (see Figure 4.1):

(1) Starting with a prescribed position x_1, a sequence of positions x_i with $i = 1, 2, \ldots$ is computed under the condition $N\,(x_i, x_{i+1}, t) = 1$.

(2) The spatial interval $[x_i, x_{i+1}[$ contains the vehicle i which is placed at position x_i and has the distance $s_i = x_{i+1} - x_i$ to its preceding vehicle $i + 1$.

(3) The average velocity $\bar{v} = (x_i, x_{i+1}, t)$ represents the velocity v_i of vehicle i.

Using this transformation procedure, macroscopic traffic states can be evaluated on a microscopic level.

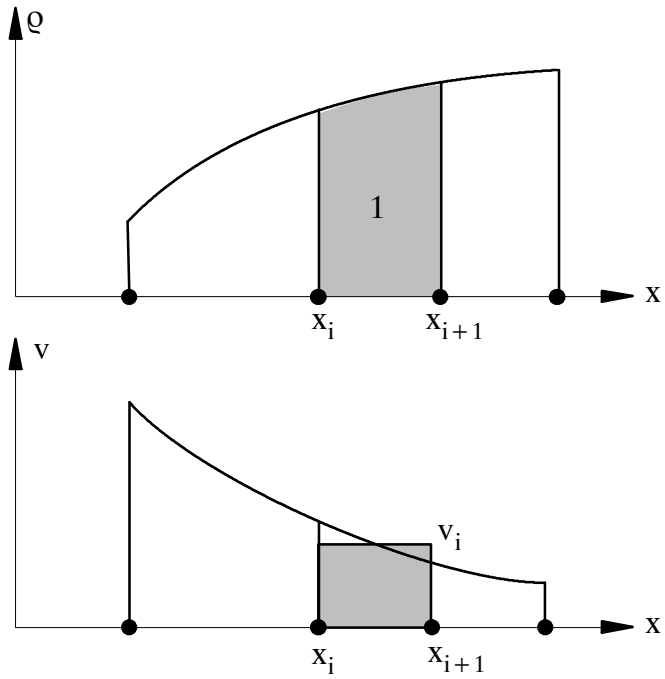


Figure 4.1: Macroscopic and microscopic traffic state

4.2 Dynamic Equations

Macroscopic traffic dynamics of single lane models are very similar to one-dimensional fluid dynamics. The governing equations are partial differential equations for the density $\varrho\,(x,t)$ and the mean velocity $v\,(x,t)$ which are called continuity equation and equation of motion.

Continuity equation : The law of the conservation of the number of vehicles leads to the continuity equation for the density $\varrho\,(x,t)$:

$$\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x}\,(\varrho\, v) = 0 \tag{4.3}$$

Equation of motion : In contrast to fluid dynamics, the law of conservation of momentum is not valid in traffic dynamics. That is why the equation of motion is formulated differently in the various existing macroscopic traffic models. Typical formulations of the equation of motion for the mean velocity v (x, t) are:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{V - v}{\tau} \tag{4.4}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{V - v}{\tau} - \frac{1}{\varrho}\,\frac{\partial p}{\partial x} \tag{4.5}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{V - v}{\tau} - \frac{c^2}{\varrho}\,\frac{\partial \varrho}{\partial x} + \frac{\mu}{\varrho}\,\frac{\partial^2 v}{\partial x^2} \tag{4.6}$$

V	equilibrium velocity
p	traffic pressure
c, μ, τ	parameters

The left side of the equation of motion represents the total acceleration of the traffic stream which consists of the local acceleration $\partial v/\partial t$ and the convective acceleration $v\ \partial v/\partial x$. The right side of the equation of motion represents the driving forces of the traffic stream. The meaning of the different terms for the driving forces are explained below.

Adaption : The term $(V - v) / \tau$ is called the adaption term or relaxation term. It is assumed that the current velocity $v\ (x, t)$ is adapted to a prescribed equilibrium velocity V within a certain time τ. The equilibrium velocity V, which depends at least on the density, and the adaption time are the most important parameters of all macroscopic traffic models. The equation of motion (4.4) is equivalent to the dynamic equation (3.2) of the microscopic traffic model.

Pressure : The term $(1/\varrho)\ \partial p/\partial x$ is called pressure term or anticipation term. Different approaches are used for the pressure. A 'hydrodynamic' approach assumes that the pressure is proportional to the density.

$$p = c^2\ \varrho \tag{4.7}$$

c propagation velocity

A 'gasdynamic' approach [4] assumes that the pressure is proportional to the density and to the square of the velocity.

$$p = A\ (\varrho)\ \varrho\ v^2 \tag{4.8}$$

$A\ (\varrho)$ density dependent proportionality factor

The pressure term reflects the traffic behaviour that the velocity increases with locally decreasing pressure and vice versa. The equation of motion (4.5) is similar to the Euler equation in fluid dynamics.

Viscosity : The term $(\mu/\varrho)\ \partial^2 v/\partial x^2$ with the viscosity parameter μ is called viscosity term which reflects some observations of viscous traffic flow in reality. The equation of motion (4.6) is similar to the Navier-Stokes equation in fluid dynamics.

Conservative form : The above dynamic equations can be reformulated in the conservative form with the density $\varrho\ (x, t)$ and the flow $q\ (x, t)$ as primary variables. Using the flow relationships

$$q = \varrho\ v \qquad\qquad Q = \varrho\ V \tag{4.9}$$

the corresponding conservative forms of the continuity equation (4.3) and the equation of motion (4.5) are:

$$\frac{\partial \varrho}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{4.10}$$

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^2}{\varrho} + p\right) = \frac{Q - q}{\tau} \tag{4.11}$$

The conservative form of the dynamic equations has advantageous properties in numerical analysis.

4.3 Equilibrium State

Equilibrium state : The density $\varrho = \text{const}$ and the mean velocity $v = V$ are solutions of the dynamic equations and represent an equilibrium state. All vehicles are moving with the same velocity V and the same distance $s = 1/\varrho$ to their predecessors. In microscopic modelling, the velocity-distance relationship is specified by a function V (s).

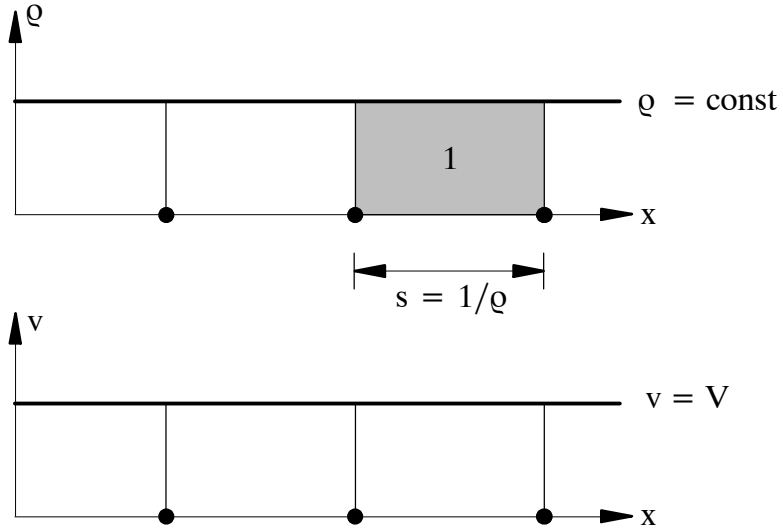


Figure 4.2: Equilibrium state

Equilibrium velocity and flow : The mean velocity of an equilibrium state is called equilibrium velocity and is specified by a function $V(\varrho)$ which decreases monotonously from $V(0) = V_0$ to $V(\varrho_{max}) = 0$.

$$V = V(\varrho) \qquad 0 \le V \le V_0 \qquad \frac{dV}{d\varrho} \le 0 \tag{4.12}$$

The mean flow of an equilibrium state is called an equilibrium flow and is defined as a function $Q = \varrho V$ which has one maximum value Q_{max}.

$$Q = \varrho V \qquad 0 \le Q \le Q_{max} \tag{4.13}$$

Unfortunately, the equilibrium velocity or the equilibrium flow can not be determined from measurement data because equilibrium states do not occur in real traffic. That is why in the literature different approaches for the equilibrium velocity are proposed. Some typical approaches are shown in Figure 4.3.

Stability : An equilibrium state is stable, if small deviations from the equilibrium density or the equilibrium velocity decrease with time evolution. The stability condition can be determined using the methods of linear stability theory. For the continuity equation (4.4) and

the Navier-Stokes-like equation of motion (4.6), the stability depends on the derivative $V'(\varrho)$ of the equilibrium velocity and the propagation velocity c:

$$\text{stability} \quad |V'(\varrho)| < c/\varrho \tag{4.14}$$

If the stability condition for an equilibrium state is not satisfied, small perturbations are producing moving jams.

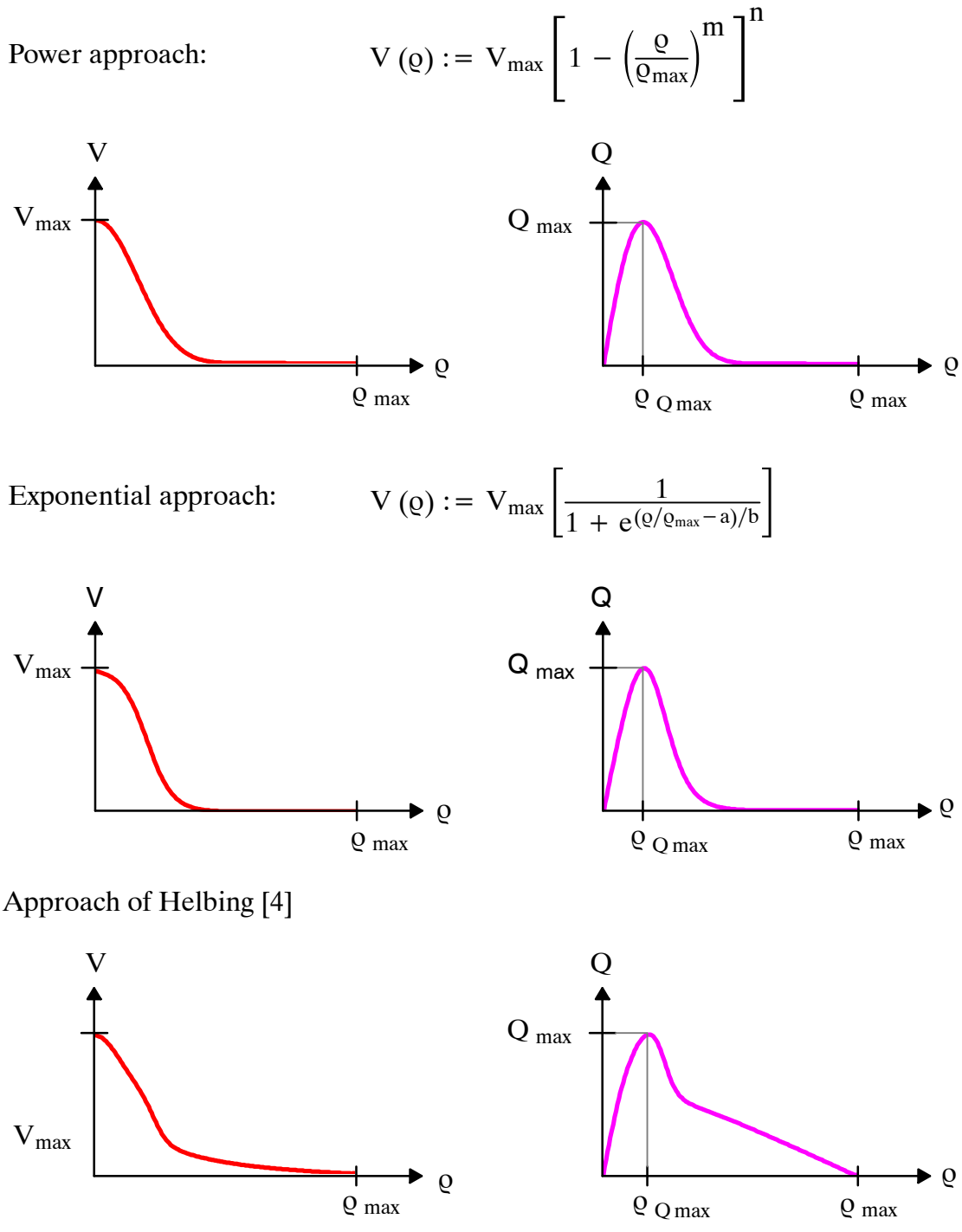


Figure 4.3 Equilibrium velocity and fundamental diagram

4.4 Initial and Boundary Conditions

Initial conditions : The space dependent initial conditions describe the traffic state on a single lane at the time $t = 0$.

$$\varrho\,(x, 0) = \bar{\varrho}\,(x) \qquad v\,(x, 0) = \bar{v}\,(x) \tag{4.15}$$

If a closed lane without boundaries is considered, the results of traffic simulations depend significantly on the specified initial conditions. For an open single lane however, the initial conditions influence the results of traffic simulations only for a short time period.

Boundary conditions : The time dependent boundary conditions describe the traffic states at the upstream boundary $x = 0$ and the downstream boundary $x = L$ of an open single lane of length L. There are different options for the specification of boundary conditions. Usually Dirichlet and von Neumann boundary conditions are used:

(1) The Dirichlet boundary conditions assume that the traffic states at the boundaries are given.

$$\begin{aligned} \varrho\,(0, t) &= \varrho_0\,(t) & v\,(0, t) &= v_0\,(t) \\ \varrho\,(L, t) &= \varrho_L\,(t) & v\,(L, t) &= v_L\,(t) \end{aligned} \tag{4.16}$$

(2) The homogeneous von Neumann boundary conditions assume that the traffic states at the boundaries remain unchanged.

$$\begin{aligned} \frac{\partial \varrho}{\partial x}\,(0, t) &= 0 & \frac{\partial v}{\partial x}\,(0, t) &= 0 \\ \frac{\partial \varrho}{\partial x}\,(L, t) &= 0 & \frac{\partial v}{\partial x}\,(L, t) &= 0 \end{aligned} \tag{4.17}$$

Dirichlet boundary conditions are suitable for the simulation of real traffic with measured traffic data at both boundaries. Homogeneous von Neumann boundary conditions can be used if the traffic states outside the lane are not of interest. The boundary conditions must be consistent with the solutions of the governing equations. Inconsistent boundary conditions lead to numerical instabilities.

5 NUMERICAL SIMULATION METHODS

The numerical solution of the dynamic equations of macroscopic traffic models is a particular difficult task because the formation of traffic jams is associated with steep gradients of the density and velocity distributions. Numerical methods require a discretization of the space-time domain and a suitable approximation of the density and velocity distribution. The development of numerical simulation models is based on the methods of finite volumes, finite differences or finite elements in connection with explicit or implicit time integration methods. Three numerical simulation models, which are selected from the various existing models, are described briefly in the following sections.

5.1 Finite Volume Method

The method of finite volumes allows the formulation of the governing equations in a discrete form using conservation principles. Hilliges [5] developed a simple but efficient discrete model on this basis.

Discrete model : The lane is subdivided into finite volume cells of length Δx. For each cell i, the density $\varrho\,(i, t)$ and the velocity v (i, t) are introduced as primary variables.

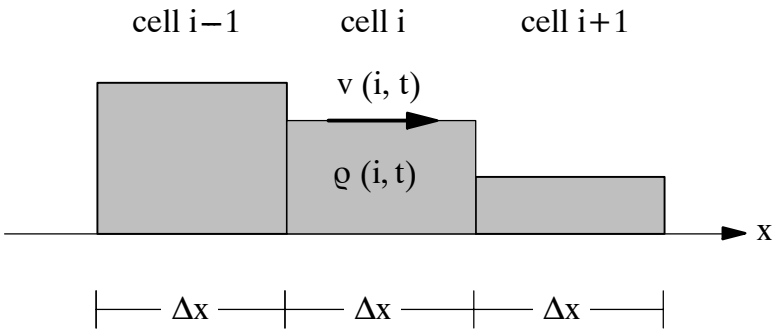


Figure 5.1 : Finite volume cells

The continuity equation and the equation of motion are specified in a discrete form as follows:

$$\frac{\partial \varrho\,(i,t)}{\partial t} = \frac{\varrho\,(i-1,t)\;\; v\,(i,t) - \varrho\,(i,t)\;\; v\,(i+1,t)}{\Delta x} \tag{5.1}$$

$$\frac{\partial v\,(i,t)}{\partial t} = v\,(i,t)\;\frac{v\,(i-1,t) - v\,(i+1,t)}{2\,\Delta x} + \frac{V\,(\varrho\,(i,t)) - v\,(i,t)}{\tau} \tag{5.2}$$

It is assumed in the continuity equation (5.1) that the vehicles of cell i−1 enter the cell i with the velocity v (i, t) and the vehicles of cell i leave cell i with the velocity v (i+1, t). This assumption reflects the anticipating behaviour of the drivers. The equations in a discrete form satisfy the condition that the density and the velocity of each cell is always positive or zero.

Continuous approximation : The equations of the discrete model are transformed into partial differential equations by Taylor approximations. Let x be the coordinate of the

centre of cell i. Then the Taylor approximations for the density and the velocity lead to the following partial differential equations :

$$\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x}(\varrho v) = \frac{\Delta x}{2}\left(v\frac{\partial^2 \varrho}{\partial x^2} - \varrho\frac{\partial^2 v}{\partial x^2}\right) + O(\Delta x^2) \tag{5.3}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{V(\varrho) - v}{\tau} + O(\Delta x^2) \tag{5.4}$$

Neglecting terms of quadratic or higher order in Δx, equation (5.3) represents the continuity equation (4.3) extended by an additional diffusion term which is proportional to Δx. Equation (5.4) is equivalent to the equation of motion (4.4).

Numerical simulation : The equations of the discrete model are solved numerically using an explicit Euler method for the time integration.

$$\begin{aligned} \varrho(i, t + \Delta t) &= \varrho(i, t) + \Delta t \frac{\partial \varrho(i, t)}{\partial t} \\ v(i, t + \Delta t) &= v(i, t) + \Delta t \frac{\partial v(i, t)}{\partial t} \end{aligned} \tag{5.5}$$

The stability of the numerical simulation depends on the parameter Δx, Δt and τ. The cell length Δx should not be less than 100 m, because otherwise the implicit diffusion term is too low and the model becomes unstable. The time step Δt must be chosen in such a way that the Courant condition $v \leq \Delta x/\Delta t$ for the explicit time integration is satisfied. Typical values for the adaption time τ are chosen in the range from 3.0 to 7.5 sec.

5.2 Finite Difference Method

The methods of finite differences approximate the derivatives of differential equations by differences. The Keller-box scheme is a finite difference method of high accuracy for non-linear partial differential equations of first order. Kerner and Kohnhäuser [6] developed a numerical method for the solution of the continuity equation and the Navier-Stokes-like equation on the basis of the Keller-box scheme.

Dynamic equations : The dynamic equations are the continuity equation (4.3) and the Navier-Stokes-like equation of motion (4.6). An additional equation for the spatial derivative $\partial v/\partial x$ is introduced so that the governing partial differential equations are of first order.

$$\begin{aligned} f_1 &:= \frac{\partial \varrho}{\partial t} + v\frac{\partial \varrho}{\partial x} + \varrho\frac{\partial v}{\partial x} = 0 \\ f_2 &:= \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} - \frac{c^2}{\varrho}\frac{\partial \varrho}{\partial x} + \frac{\mu}{\varrho}\frac{\partial w}{\partial x} - \frac{V - v}{\tau} = 0 \\ f_3 &:= \frac{\partial v}{\partial x} - w = 0 \end{aligned} \tag{5.6}$$

Finite difference approximation : The continuous functions $\varrho\,(x,t)$, $v\,(x,t)$ and $w\,(x,t)$ are computed at discrete grid points (x_i, t_j) in the space-time domain and denoted by $\varrho_{i,j}$, $v_{i,j}$ and $w_{i,j}$. Usually a regular grid with a spatial increment Δx and a time step Δt is chosen.

$$x_i = i \cdot \Delta x \qquad i = 0, 1, ..., n$$
$$t_j = j \cdot \Delta t \qquad j = 0, 1, ...$$

Figure 5.2 shows a grid box with $x_{i-1} \le x \le x_i$ and $t_{j-1} \le t \le t_j$. The value and the partial derivatives of a function $u \in \{\varrho, v, w\}$ at the midpoint m of the box are calculated approximately from the function values at the box vertices by the following formulas:

$$u'_m := \left(\frac{\partial u}{\partial x}\right)_m \qquad \dot{u_m} := \left(\frac{\partial u}{\partial t}\right)_m$$
$$u_m = (u_{i,j} + u_{i-1,j} + u_{i,j-1} + u_{i-1,j-1}) / 4$$
$$u'_m = (u_{i,j} - u_{i-1,j} + u_{i,j-1} - u_{i-1,j-1}) / (2 \cdot \Delta x)$$
$$\dot{u_m} = (u_{i,j} + u_{i-1,j} - u_{i,j-1} - u_{i-1,j-1}) / (2 \cdot \Delta t) \qquad (5.7)$$

The substitutions of the above approximations into the partial differential equations lead to finite difference equations for the midpoint of the box:

$$f_{1,m} := \dot{\varrho_m} + v_m \varrho'_m + \varrho_m v'_m = 0$$
$$f_{2,m} := \dot{v_m} + v_m v'_m - c^2 \varrho'_m / \varrho_m + \mu w'_m / \varrho_m - (V(\varrho_m) - v_m) / \tau = 0$$
$$f_{3,m} := v'_m - w_m = 0 \qquad (5.8)$$

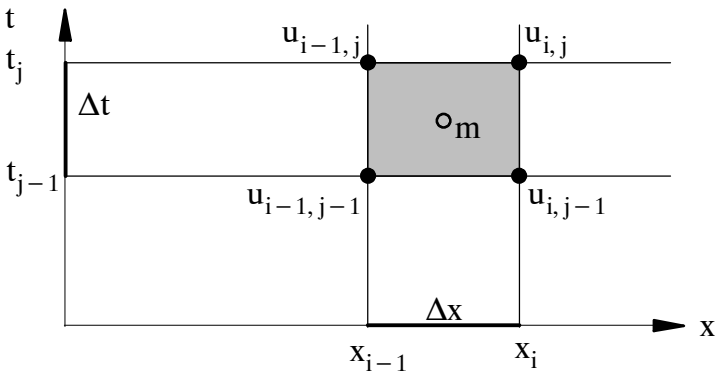


Figure 5.2 : Box scheme for finite difference approximations

The three necessary boundary conditions for ϱ, v, w at time t_j and the finite difference equations for the midpoints of all boxes with $t_{j-1} \le t \le t_j$ form a system of nonlinear equations. If the variables $\varrho_{i,j-1}$, $v_{i,j-1}$, $w_{i,j-1}$ for i = 0, 1, ..., n at time t_{j-1} are known, the variables $\varrho_{i,j}$, $v_{i,j}$, $w_{i,j}$ for $i = 0, 1, ..., n$ at time t_j are computed by solving the nonlinear equations. It is useful to arrange the nonlinear equations for the unknown variables in the following block by block form:

$$\mathbf{f}(\mathbf{u}) = \mathbf{0} \tag{5.9}$$

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_n \end{bmatrix}$$

boundary conditions

finite difference equations

$$\mathbf{u}_i = \begin{bmatrix} \varrho_{i,j} \\ v_{i,j} \\ w_{i,j} \end{bmatrix} \qquad \mathbf{f}_m = \begin{bmatrix} f_{1,m} \\ f_{2,m} \\ f_{3,m} \end{bmatrix}$$

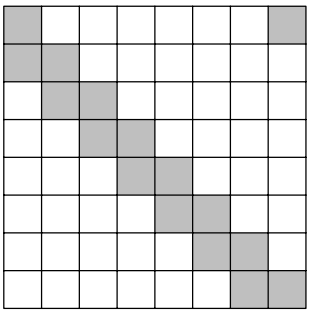
Nonlinear equation system : The nonlinear equations are linearized and solved iteratively using the Newton procedure.

$$\mathbf{J}(\mathbf{u}_k) \cdot \Delta \mathbf{u}_{k+1} = -\mathbf{f}(\mathbf{u}_k) \qquad \text{with} \qquad \mathbf{J}(\mathbf{u}) = \partial \mathbf{f} / \partial \mathbf{u}$$

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \Delta \mathbf{u}_{k+1} \tag{5.10}$$

A linear equation system with the Jacobian matrix **J** has to be solved in each iteration step k. The Jacobian matrix contains the partial derivatives of the nonlinear equations with respect to the unknown variables and has the following block structure:

$\mathbf{J} =$



The special structure of the matrix **J** allows the implementation of efficient algorithms for the solution of the linear equation system (5.10). However, the matrix **J** is not diagonal dominant. Hence, algorithms based on Gauss elimination without rearrangements of the structure are not suitable. Algorithms based on orthogonal transformations (Givens, Householder) are numerically very stable and can be used effectively for the solution of the linear equation system.

Numerical simulation : The finite difference approximation for the solution of the dynamic equations has a consistency order $O(\Delta x^2, \Delta t^2)$ and is numerically stable. The spatial increment Δx and the time step Δt should be chosen depending on the application. However, the numerical simulation requires a considerable computational effort because a nonlinear equation system has to be solved in each time step.

5.3 Finite Element Method

Finite element methods are widely used for numerical computations in fluid dynamics. Rose and Milbradt [7] developed a finite element method for traffic dynamics based on the continuity equation and the Navier-Stokes-like equation of motion.

Dynamic equations : The continuity equation (4.3) and the Navier-Stokes-like equation of motion (4.6) are formulated in vector notation as follows:

$$\mathbf{I}\,\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}\,\frac{\partial \mathbf{u}}{\partial x} - \mathbf{B}\,\frac{\partial^2 \mathbf{u}}{\partial x^2} = \mathbf{f} \tag{5.11}$$

$$\mathbf{u} = \begin{bmatrix} \varrho \\ v \end{bmatrix} \quad \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} v & \varrho \\ c^2/\varrho & v \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & \mu/\varrho \end{bmatrix} \quad \mathbf{f} = \begin{bmatrix} 0 \\ (V-v)/\tau \end{bmatrix}$$

Finite element approximation : The lane is subdivided into finite elements. At each node i with the coordinate x_i the time dependent nodal vector $\mathbf{u}_i\,(t)$ is introduced as a primary variable which contains the density $\varrho_i\,(t)$ and the velocity $v_i\,(t)$. The solution vector $\mathbf{u}\,(x, t)$ is approximated by a function $\mathbf{u}_f\,(x, t)$ which is a linear combination of the nodal vectors $\mathbf{u}_i\,(t)$ and the normalized triangular shape function $\Phi_i\,(x)$.

$$\mathbf{u}_f\,(x,t) = \sum_{i=1}^{n} \Phi_i\,(x)\,\mathbf{u}_i\,(t)$$

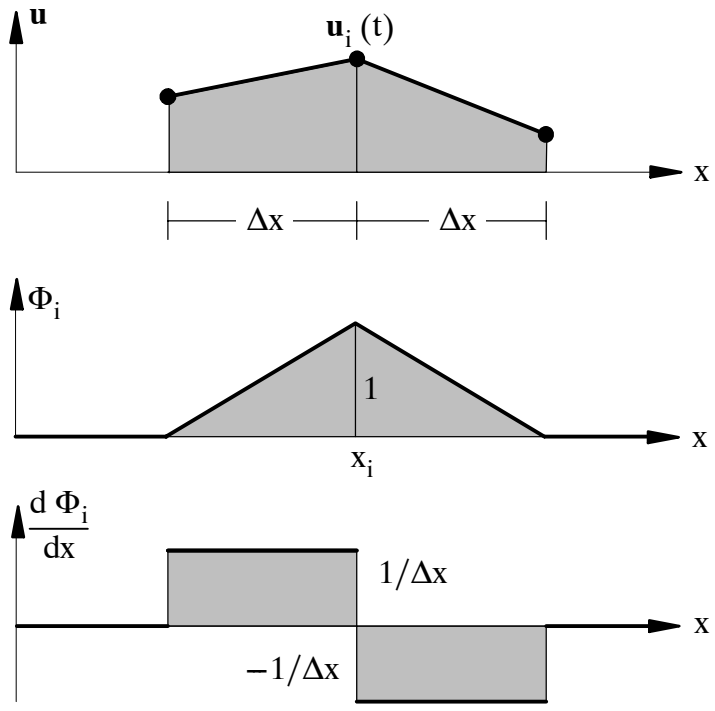


Figure 5.3 : Finite element approximation

The approximation $\mathbf{u}_f\,(x, t)$ is substituted into the partial differential equation system (5.11). The result of this substitution is the residual vector $\mathbf{r}\,(x, t)$.

$$\mathbf{r}\,(x,t) = \sum_{i=1}^{n} \left(\mathbf{I}\,\Phi_i \frac{d\mathbf{u}_i}{dt} + \mathbf{A}\,\frac{d\Phi_i}{dx}\,\mathbf{u}_i - \mathbf{B}\,\frac{d^2\Phi_i}{dx^2}\,\mathbf{u}_i \right) - \mathbf{f} \tag{5.12}$$

The nodal vectors $\mathbf{u}_i\,(t)$ are determined on the basis of weighted residuals using a Petrov-Galerkin method with an upwind parameter α.

$$\int_0^L \left[\mathbf{I}\,\Phi_i + \alpha\,\mathbf{A}\,\frac{d\Phi_i}{dx} \right] \mathbf{r}\,(x,t)\,dx = \mathbf{0} \qquad\qquad i = 1, ..., n \tag{5.13}$$

The Standard-Galerkin method ($\alpha = 0$) tends to unrealistic oscillations and even instabilities, if the convective term **A** is dominant compared to the diffusion term **B** in the dynamic equations. The upwind term with $\alpha > 0$ of the Petrov-Galerkin method should compensate the oscillations and ensure the stability of the finite element approximation.

Upwind parameter : The upwind parameter α is known and theoretically proved for the one-dimensional convection-diffusion equation with a scalar function u (x, t), a constant convection coefficient a and a constant diffusion coefficient b.

$$\alpha = \frac{\Delta x}{|a|}\,(\coth Pe - 1/Pe) \qquad\qquad Pe = \frac{|a|}{|b|}\,\Delta x \tag{5.14}$$

Pe = Peclet number

This result can be overtaken to the dynamic equations (5.11) if representative values a and b can be determined for the convection matrix **A** and the diffusion matrix **B**. It seems to be a suitable approach to use the largest eigenvalues of **A** and **B**.

$$|a| = v + c \qquad\qquad |b| = \mu/\varrho \tag{5.15}$$

This approach for the upwind parameter leads to good numerical results, but has to be proved theoretically.

Numerical simulation : The governing equations for the nodal vectors $\mathbf{u}_i\,(t)$ determined by the Petrov-Galerkin method form a system of ordinary differential equations.

$$\mathbf{M}\,\frac{d\,\mathbf{u}}{d\,t} = \mathbf{C}\,\mathbf{u} + \mathbf{f} \qquad\qquad \mathbf{u}^T = \left[\mathbf{u}_1 \ldots \mathbf{u}_n\right] \tag{5.16}$$

These equations are solved numerically using an explicit method for the time integration. Efficient algorithms for the numerical solution can be implemented which reduce the matrix **M** to a diagonal matrix and control the time step automatically using an empirical condition $|a| \le \Delta x/\Delta t$.

5.4 Numerical Perturbation Analysis

Numerical traffic simulations are investigated on the basis of a numerical perturbation analysis with respect to the formation of traffic jams. For this purpose, a closed single lane is considered. It is assumed that the initial traffic state is an equilibrium state with a prescribed density ϱ_e and a small local perturbation of the uniform density distribution. The properties of the jam formations are studied in detail depending on the density ϱ_e.

Figure 5.4 shows some typical results of a numerical perturbation analysis for macroscopic traffic models based on the continuity equation and the Navier-Stokes-like equations using the finite element method [7]. The equilibrium is stable for free traffic with $\varrho_e < 15$ veh./km and for dense traffic with $\varrho_e > 60$ veh./km. In the stable case the perturbations of the equilibrium state decreases with increasing time. The equilibrium is unstable for congested traffic. There are three different phenomena of congested traffic with respect to jam formations: simple jam, stop and go traffic and wide jam. The jams move backward with a velocity of about -15 ± 5 km/h. The different phenomena of jam formations can also be observed in real traffic on highways.

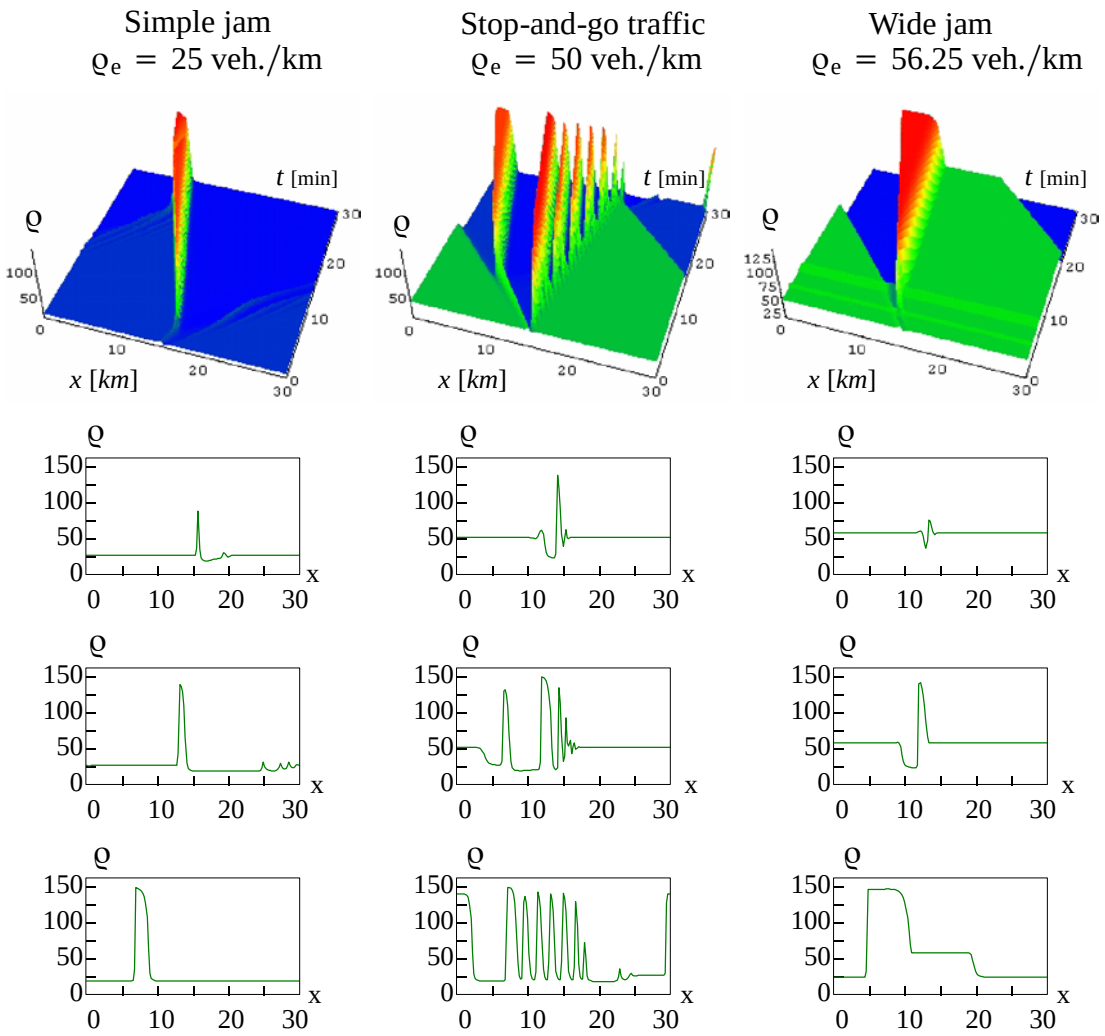


Figure 5.4 : Density evolutions in the space-time diagram
and spatial density distributions at time t = 3, 11, 30 min

6 TRAFFIC MEASUREMENTS AND SIMULATIONS

Traffic scenario : Figure 6.1 shows a schematic diagram of a 9 km section of the A5 highway in Germany with three lanes, one off ramp, one on ramp and ten intersections Q79–Q89 with installed detector equipments. Measurement data have been made available by the Board of Road and Transportation in the state Hessen. A locally fixed jam was observed on Tuesday 27 January 2001 which was caused by an accident behind the intersection Q89. This traffic scenario is chosen for a comparison of measurements and simulations.

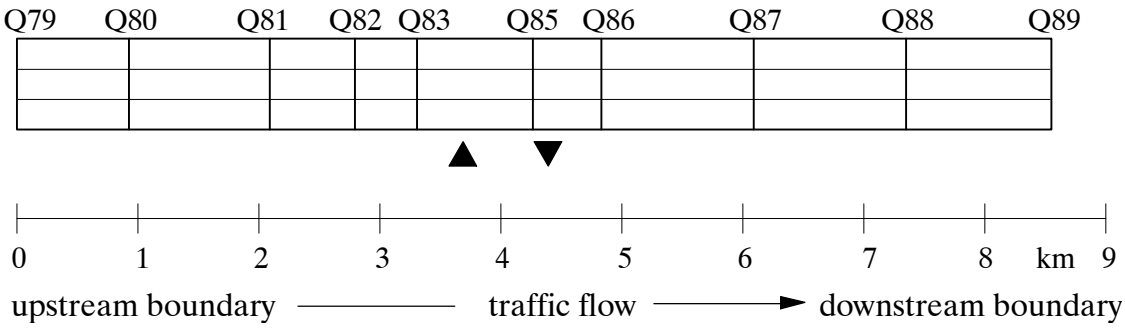


Figure 6.1: Section of the A5 highway

Macroscopic single-lane model : The traffic scenario is simulated using a single-lane model. The initial conditions, the boundary conditions and the ramp flows are calculated from the measurement data. The begin of the simulation is chosen in such a way that the initial state is a state of free traffic. The time dependent density and velocity at the upstream boundary and the time dependent velocity at the downstream boundary are specified on the basis of the detector data at the intersections Q79 and Q89. The boundary conditions are shown in figure 6.2. The low densities and the high velocities at the upstream boundary represent free traffic. The low velocities at the downstream boundary indicate a traffic jam.

Model parameter : The method of Hilliges (see section 5.2) is used for the solution of the dynamic equations of the single-lane model. The discretization parameters of the space-time domain depend on the accuracy and the stability of the method and are chosen as follows:

$$\Delta x = 100 \text{ m} \qquad \Delta t = 1 \text{ sec} \tag{6.1}$$

The equilibrium velocity-density function and the adaption time τ are the most important parameters of single-lane models and depend significantly on the traffic scenario. For this scenario, the equilibrium velocity-density function of Helbing is used which is described by a set of eight parameters. The essential parameters are the maximum velocity and the maximum density which are chosen as follows:

$$V_{max} = 125 \text{ km/h} \qquad \varrho_{max} = 135 \text{ veh./km} \tag{6.2}$$

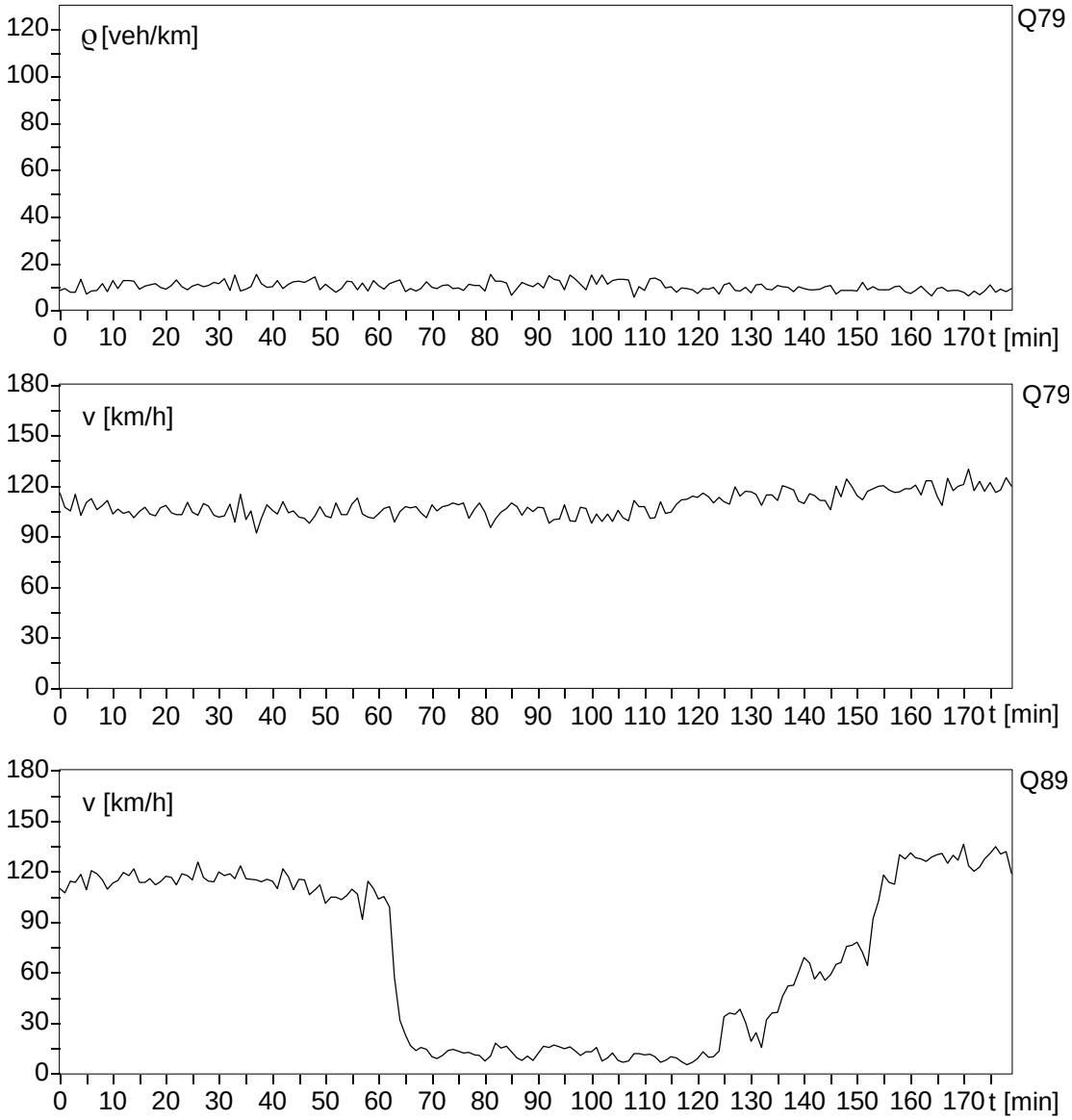


Figure 6.2: Upstream and downstream boundary conditions

The other parameters are taken from the literature [4]. The adaption time τ is determined by variation. Good simulation results are obtained for $\tau = 2$ sec.

Comparison : Figure 6.3 shows the simulated and measured velocities at the intersections Q89−Q86 depending on the time. A good agreement between the simulated and measured velocity profiles could be achieved with respect to the upstream and downstream front of the jam. The upstream front moves backward with a velocity of about −4,5 km/h which depends on the inflow into the jam. The downstream front moves faster backward than the upstream front with a velocity of about −12 km/h and an outflow from the jam which is approximately constant.

Figure 6.3 shows the corresponding density profiles at the intersections Q89−Q86. The simulated density profiles and the density profiles calculated from local measurements differ significantly in the jam region. As mentioned in section 2, the densities calculated from local measurements are not reliable if the corresponding velocities and flows are low.

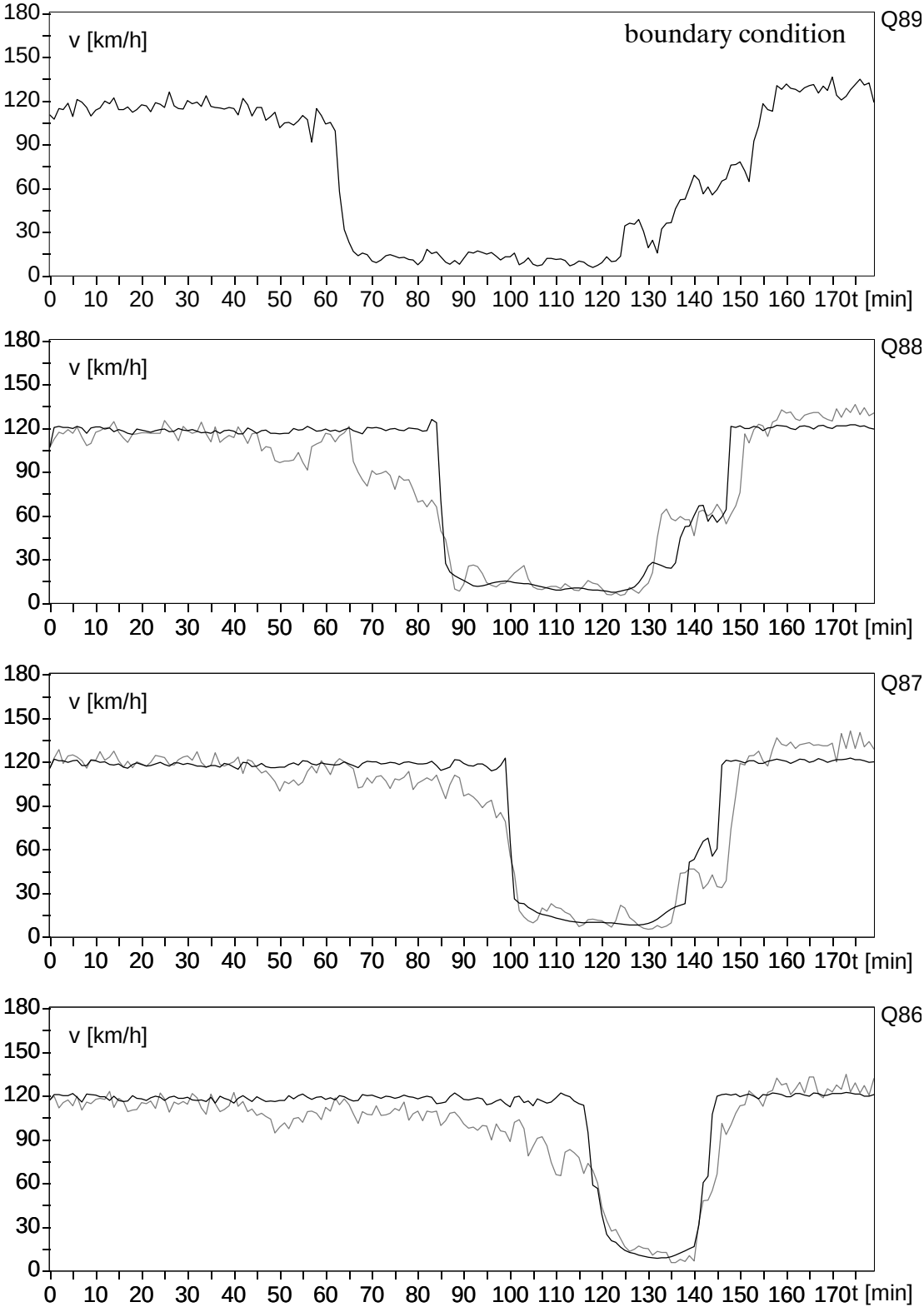


Figure 6.3: Simulated and measured velocities
simulation measurement

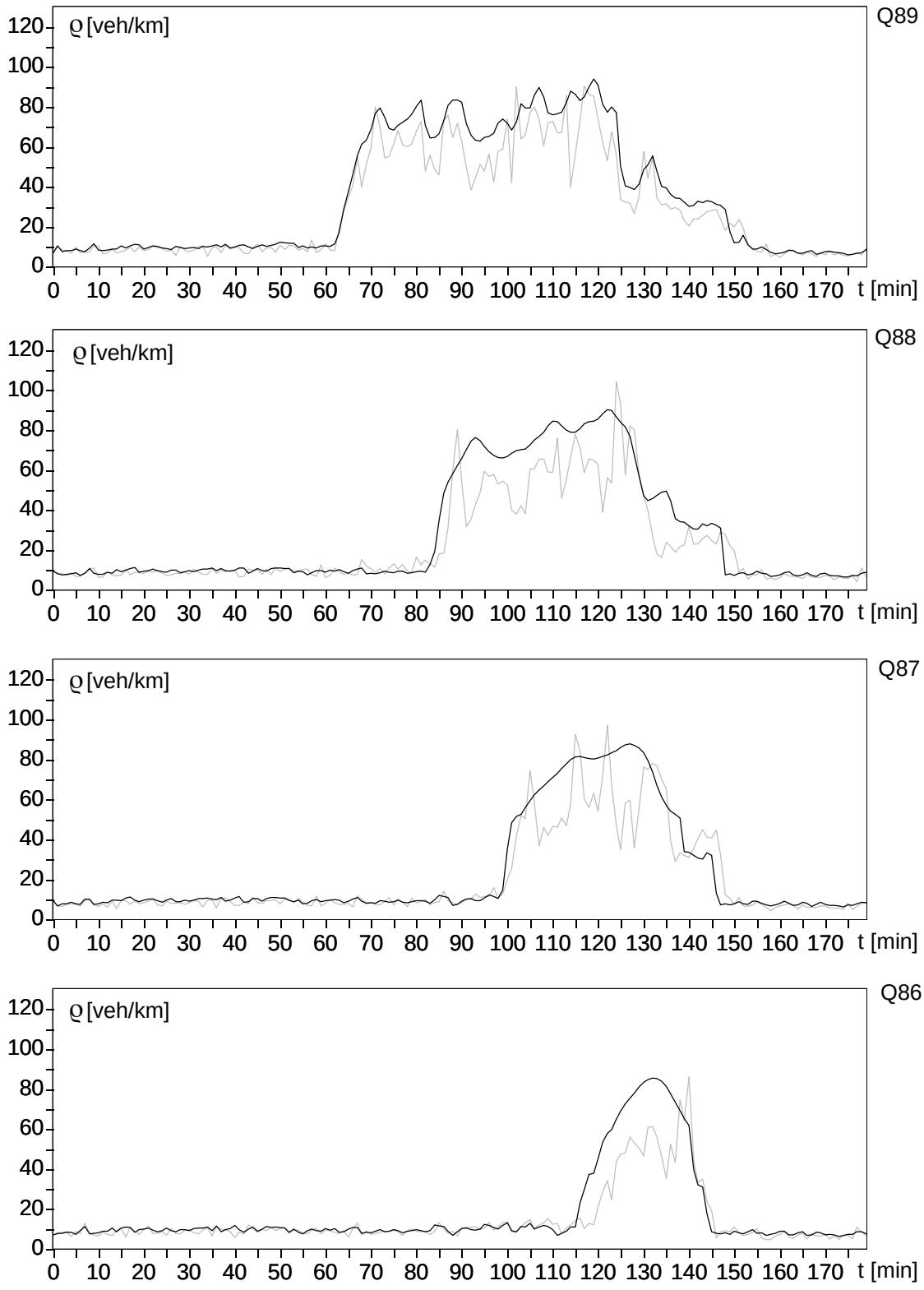


Figure 6.4: Simulated and measured densities
simulation measurement

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