GENERALIZED STABILIZATION TECHNIQUES IN COMPUTATIONAL FLUID DYNAMICS

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ABSTRACT

A procedure based on the Galerkin/least-squares FEM for advective-diffusive equations is generalized to cover the FVM and FDM. The numerical realization is performed as a local error correction in three steps. In the first step a global approximation is calculated. In the second step the local residua are determined and then used in combination with the stabilization parameter to correct the global approximation. Two numerical test cases of a transport problem and dam-break induced flow were carried out. These tests show the efficient applicability of the presented generalized stabilization techniques.

Keywords: FEM, FVM, FDM, Generalized Stabilization Techniques (GST), Dam-break

1. INTRODUCTION

Many of the physical, technical and natural phenomenons can be modelled by the Advection-Diffusion Equation. Examples are the transport of a solute, shallow water, morphodynamics as well as traffic flow etc. The transient Advection-Diffusion Equation is given, according to the approximating numerical method, in the direction independent form in Eq. 1, the direction dependent form in Eq. 2, and the operator form in Eq. 3 and Eq 4.

$$\frac{\partial U}{\partial t} + div(U\boldsymbol{u}) = div(\gamma\, grad\, U) + S \tag{1}$$

$$\frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x} + v\frac{\partial U}{\partial y} = \gamma\frac{\partial^2 U}{\partial x^2} + \gamma\frac{\partial^2 U}{\partial y^2} + S \tag{2}$$

$$\frac{\partial U}{\partial t} + LU + S = 0 \tag{3}$$

where
$$L \equiv A_i\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial}{\partial x_j}\right) \tag{4}$$

In the previous equations U is the unknown variable of the problem, $\boldsymbol{u}$ is the velocity vector with the x- and y- components u and v, γ is the diffusion coefficient and S is the source and sink. In Eq. 3 and 4 L is the differential operator, A_i is the i^{th} Euler Jacobian matrix and K_{ij} is the diffusivity matrix.

The most commonly used numerical methods to approximate the solution of the transient Advection-Diffusion Equation are the Finite Difference Method (FDM), the Finite

Volume Method (FVM) and the Finite Element Method (FEM). The essential differences between these methods can be illustrated as in Figure 1.

Using the FDM the differential equation will be approximated by a difference equation. This is a very easy method with an equidistant structured calculating mesh. The FEM approximates the solving space of the PDE; and the unknown function is interpolated by a set of basis functions, that will be helpful in calculating the derivatives. It is apparently more complex and has more flexibility decomposing the solution domain. The FVM starts with the integration of the PDE over a control volume. By using the divergence theorem the integral over the volume will be substituted against an integral over the control volume's boundary. It is simpler than the FEM at the first sight and still has the flexibility of arbitrarily decomposing the solution domain but with special volume decomposition requirements.

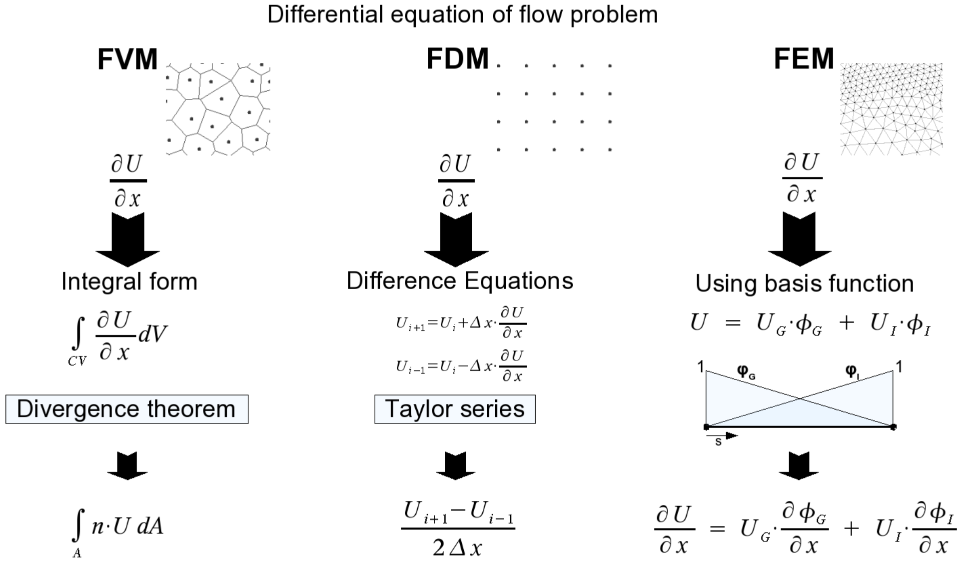

Figure 1: The most commonly used numerical methods and their essential differences

The mere standardized application of these numerical methods in solving the Advection-Diffusion Equation results in a nonphysical oscillated approximation of the advection dominant PDE and an instable numerical method.

Stabilising these methods is an important issue to grantee the applicability of a numerical method to real world problems. The Galerkin/least-squares finite element method overcomes the instability resulting from only applying the Galerkin method by introducing a mesh dependent least square term see Hughes (1989). This approach was adapted by Milbradt (2002) and formulated as a local residua correction. The computation of suitable stabilization parameters depends on the differential operators of the problem and the mesh parameters.

The Generalized Stabilisation Techniques (GST) presented in this paper broaden the stabilization formalism used in the Galerkin/least-squares FEM to cover both the FVM and the FDM. In the next sections the basics of the GST are illustrated; the formulation for the FEM, FVM and FDM are derived, the choice of a suitable stabilisation parameter is discussed and the numerical application through one dimensional transport problem simulation and two dimensional dam-break simulation is demonstrated.

2. BASIC IDEA

A one dimensional example of Eq. 1 without the diffusion term, which does not contribute to the instability of the numerical method, will be used to demonstrate the basic idea of the GST.

After each time step of the global standard numerical approximation of the solution of

$$\frac{\partial U}{\partial t} + div(U\mathbf{u}) - S = 0 \tag{5}$$

the averaged local residua for each element is calculated according to

$$\bar{\epsilon}_U = \frac{\left(\frac{\partial U}{\partial t} + div(U\mathbf{u}) - S\right)_G + \left(\frac{\partial U}{\partial t} + div(U\mathbf{u}) - S\right)_I}{2} \quad , \tag{6}$$

see Figure 2.

This averaged local residua is then advectively transported as a ***correction*** of the time derivative in each of the contributed nodes as shown in Figure 2.

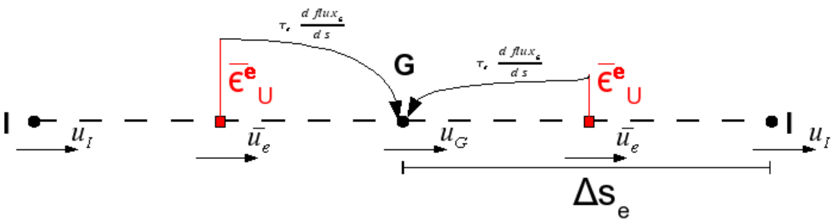

Figure 2: The basic idea of the GST

This ***correction flux*** is given as

$$\left.\frac{\partial U}{\partial t}\right|_{correction} = -\sum_e \tau_e \frac{d\, flux_\epsilon}{d\, s} \tag{7}$$

where the flux of the residua is

$$flux_\epsilon = \bar{u} \cdot \bar{\epsilon}_U \tag{8}$$

This ***correction*** is dependent on a ***correction parameter*** τ_e , that takes the shape of the spacial discretizing mesh into consideration. This parameter is given for a scalar valued problem in Eq. 9 and will be discussed later on.

$$\tau_e = \frac{1}{2} \frac{\Delta s_e}{|\bar{u}_e|} \tag{9}$$

where $|\bar{u}_e|$ the norm of the averaged velocity and Δs_e is the element size.

3. GST FORMALIZATIONS

The GST procedure can be listed in three steps, that apply to all three numerical methods, i.e. FDM, FVM or FEM, see Figure 3:

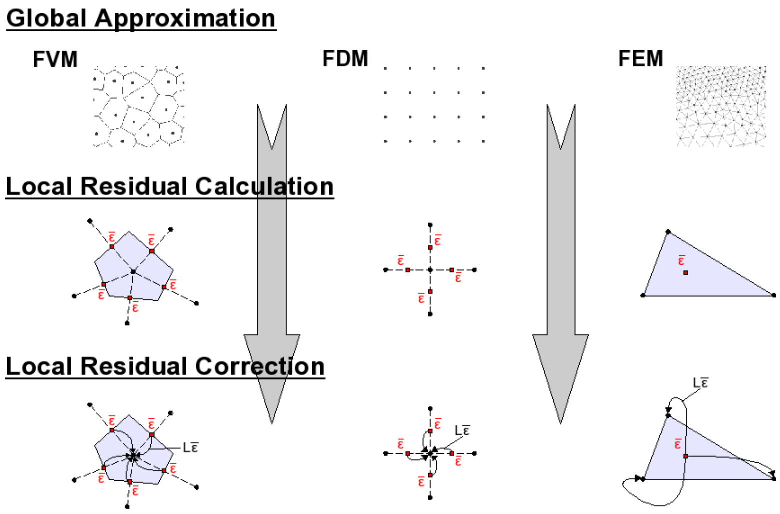

Figure 3: Sketch of the GST steps for the FVM, FDM and FEM

Applying the GST differs slightly according to the method used. The following section shows the GST formulation in a two dimensional domain for each method separately.

3.1 Stabilized FEM

In order to approximate Eq. 3 using the finite element method the domain Ω is discretized into n_{el} finite elements Ω_e .

As mentioned in the introduction the GST is a generalization of the Galerkin/least-squares finite element method (GLS). The derivation of this semi-discrete stabilized finite element approximation is carried out via the combination of a standard Galerkin approximation and the least squares approximation. This can be described roughly, for the differential equation in Eq. 3 as follows:

$$\int_{\Omega} \left(\frac{\partial U}{\partial t}+LU+S\right)\cdot w\, d\Omega \; + \; \sum_{e=1}^{n_{el}} \tau_e \int_{\Omega_e} (L\cdot w)\left(\frac{\partial U}{\partial t}+LU+S\right) d\Omega_e=0 \tag{10}$$

The first term contains the Galerkin approximation and the second term contains the least-squares stabilization which is composed of the sum of integrals over the element interiors. This approximation is called semi-discrete GLS method. In this paper a modified semi-discrete Petrov-Galerkin method is used, which is a predecessor of the GLS method, see Hughes (1989). The difference is that rather than having L operating on the weighting space, only its advective part, L_{adv} , acts there.

The ***Global Approximation*** here is achieved by using the standard Galerkin approximation, see the first term in Eq. 10.

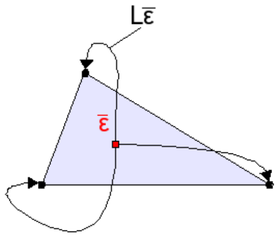

Figure 4: The Local Residual Correction of a finite element

The ***Local Residual Calculation*** and the time derivative ***Local Residual Correction*** is given in the equation

$$\int_{\Omega} \frac{\partial U}{\partial t} d\Omega_{correction} = - \sum_{e=1}^{n_{el}} \tau_e \int_{\Omega_e} (L_{adv} \cdot w)(U^G_{,t} + LU + S) d\Omega_e \quad , \tag{11}$$

where the mean element residua of the Galerkin approximation is computed and then used by applying the transportation operator and stabilization parameter for the correction of the computed time derivative.

The in such a way determined time derivatives can then be integrated over time by universal time integration procedures. The element stabilization parameter τ_e plays an important role for the stability and consistency of the approximation.

3.2 Stabilized FVM

The Approximation of Eq. 1 using the Finite Volume Method requires the discretization of the problem domain into m control volumes. The ***Global Approximation*** of the solution is achieved by using the central differencing scheme for the advection and diffusion term of Eq. 1. The time derivative of the unknown variable is thereby given in

$$\frac{\partial U}{\partial t} . V = - \sum_{i=1}^{m} \boldsymbol{n_i} . \boldsymbol{u_i} \frac{U_I + U_G}{2} A_i + \sum_{i=1}^{m} (\gamma \frac{U_I - U_G}{\Delta s_i}) A_i + S \cdot V \tag{12}$$

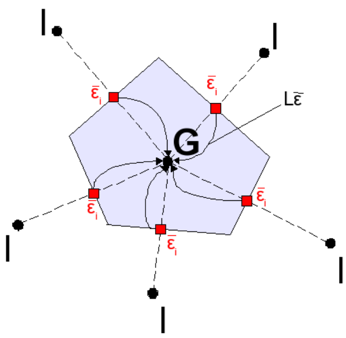

Figure 5: The Local Residual Correction of a finite volume

The ***Local Residual Calculation*** is conducted according to Eq. 6 on the neighbouring nodes across the facets of the finite volume, see Figure 5. Thus the ***Local Residual Correction*** of the computed time derivative is given, according to Eq. 7, in the form

$$\frac{\partial U}{\partial t}_{correction} = -\frac{\sum_{i}^{m} \tau_i \cdot \bar{u}_i \cdot \epsilon_{Ui} \cdot A_i}{V} \tag{13}$$

where τ_i is the correction parameter of the edge connecting the neighbouring nodes and $\bar{u}_i$ is the averaged velocity parallel to the edge.

The time derivatives can be integrated over time using universal time integration procedures.

3.3 Stabilized FDM

The Approximation of Eq. 2 using the Finite Difference Method requires the discretization of the problem domain into $n \times m$ grid nodes. The ***Global Approximation*** of the solution is achieved by using the central differencing scheme, for instance, for the advection and diffusion term of Eq. 2. The time derivative of the unknown variable is thereby given in

$$\begin{aligned} \frac{\partial U_{i,j}}{\partial t} &= -u_{i,j}\frac{U_{i+1,j}-U_{i-1,j}}{2\Delta x} - v_{i,j}\frac{U_{i,j+1}-U_{i,j-1}}{2\Delta y} \\ &+ \gamma\frac{U_{i-1,j}-2U_{i,j}+U_{i+1,j}}{\Delta x^2} + \gamma\frac{U_{i,j-1}-2U_{i,j}+U_{i,j+1}}{\Delta y^2} + S \end{aligned} \tag{14}$$

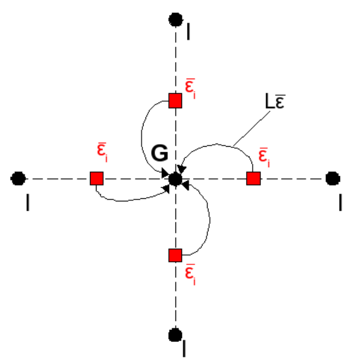

Figure 6: The Local Residual Correction of a finite difference node

The ***Local Residual Calculation*** is conducted according to Eq. 6 on the neighbouring nodes on each side in both direction x and y of the coordinate system, see Figure 6. Thus the ***Local Residual Correction*** of the computed time derivative is given, according to Eq. 7, in the form

$$\frac{\partial U}{\partial t}_{correction} = -\frac{\sum_{i=1}^{4} \tau_i \cdot \bar{u}_i \cdot \epsilon_{Ui}}{\Delta s} \tag{15}$$

where τ_i is the correction parameter of the connecting edge between the neighbouring nodes and $\bar{u}_i$ is the averaged velocity parallel to the edge.

The time derivatives can be integrated over time using universal time integration procedures.

4. STABILIZATION PARAMETER

4.1 One-Dimensional Scalar Valued Problem

In the case of a one dimensional scalar valued problem the element parameter τ_e is chosen according to the Equation

$$\tau_e = \frac{1}{2} \alpha_{opt} \frac{\Delta s_e}{|\bar{u}_e|} \tag{16}$$

where Δs_e is the element size, $|\bar{u}_e|$ is norm of the averaged velocity and α_{opt} is a Peclet number based ***Optimality Parameter*** given in

$$\alpha_{opt} = \coth(Pe) - \frac{1}{Pe} \tag{17}$$

Pe is the element Peclet number, which is given in

$$Pe = \frac{|\bar{u}_e|}{\gamma} \Delta s_e \tag{18}$$

4.2 Multi-Dimensional Scalar Valued Problem

For Application in multidimensional scalar transport problems the Euclidean norm $\|\vec{v}\|$ of the velocity vector and an element expansion h_e associated with this vector are used to define the element corrector parameter τ_e as given in

$$\tau_e := \alpha_{opt} \frac{1}{2} \frac{h_e}{\|\vec{v}\|} \tag{19}$$

4.3 Multi-Dimensional Vector Valued Problem

For multidimensional vector valued transport problems a norm $\|L_{adv}\|$ of the transport operator is used

$$\tau_e := \alpha_{opt} \frac{1}{2} \frac{h_e}{\|L_{adv}\|} \tag{20}$$

The optimality parameter α_{opt} is computed in the same way as in Eq. 17, but the element Peclet number now depends on the operator norms of the advection and diffusion differential operator

$$Pe := \frac{\|L_{adv}\| \cdot h_e}{\|L_{diff}\|} \tag{21}$$

The choice of a suitable operator norm has a significant influence on the quality of the solution. On the basis of the general definition in Kolmogorov and Fomin (1975) of the norm of continuity operators in (Euclidean) normed spaces we define the following operator norm. If the differential operator has the form

$$L_{adv}=\sum L_{adv,i}=\sum A_i \frac{\partial}{\partial x_i} \quad , \tag{22}$$

the operator norm is defined as

$$\|L_{adv}\|:=\sqrt{\sum \rho(A_i)^2} \tag{23}$$

where $\rho(A_i)$ is the spectral radius of the norm of the operator component and is calculated according to

$$\rho(A_i)=max|\lambda_j(A_i)| \tag{24}$$

i.e $\rho(A_i)$ is the largest absolute eigenvalue of the Matrix A_i.

This definition is consistent in all dimensions, starting by the one dimensional scalar valued advective diffusive problem up to more dimensional and vector valued problems. This approach for the stabilization parameter leads to very good numerical results for a large number of advection dominated problems.

5. NUMERICAL TEST CASES AND APPLICATIONS

The capabilities and characteristics of the presented generalized stabilisation techniques are demonstrated by two examples using the Finite Volume Method implementation of the GST and an explicit second order Adam-Bashforth scheme for the time integration. The first one is a classical one dimensional transport problem that assesses the correctness of the applied technique. The second is a dam break induced flow problem which displays the good stabilisation effect.

5.1 One Dimensional Transport Problem

The governing equation of the transport problem is given in Eq. 1. The transported substance concentration C is subjected to convection and diffusion in a one dimensional domain of the length $L=1m$. The size of each finite volume is $\Delta x=0.1m$ The chosen time step throughout the tests is $\Delta t=0.001s$. The transport velocity is assumed to be uniform $u=1m/s$ over the whole domain. Two tests are carried out each with a different diffusion coefficient to demonstrate the behaviour of the GST in different modes according to the variation of the Peclet number. The boundary and initial conditions are

$$C(L=0,t)=1, \ C(L=1,t)=0, \ C(x,t=0)=0.$$

The analytical solution of the steady state of this initial and boundary value problem is given in Versteeg and Malalasekera (2007). Numerical calculations using GST and the well known standard central (CD) and upwind differencing (UD) schemes see Versteeg and Malalasekera (2007) are carried out until they reach the steady state and then compared with the analytical solution.

Figure 7 shows plots of the analytical solution and numerical solutions at time $t=1s$ and after reaching the steady state for the diffusion coefficient $\gamma=0.1m^2/s$ and $Pe=1<2$ accordingly, where the transport process is diffusion dominant. The good quality of the solution using the GST can be recognized.

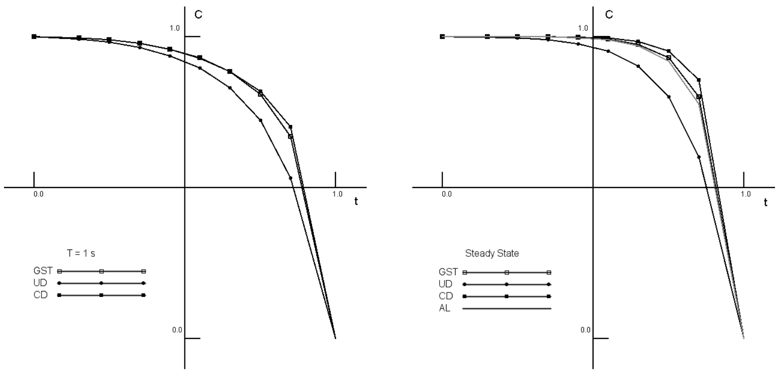

Figure 7: Analytical (AL) and numerical solutions of the diffusion dominant problem

In Figure 8 the solutions of the advection dominant transport problem with $Pe>2$ are presented for $(\gamma=0.02\ m^2/s, Pe=5)$.

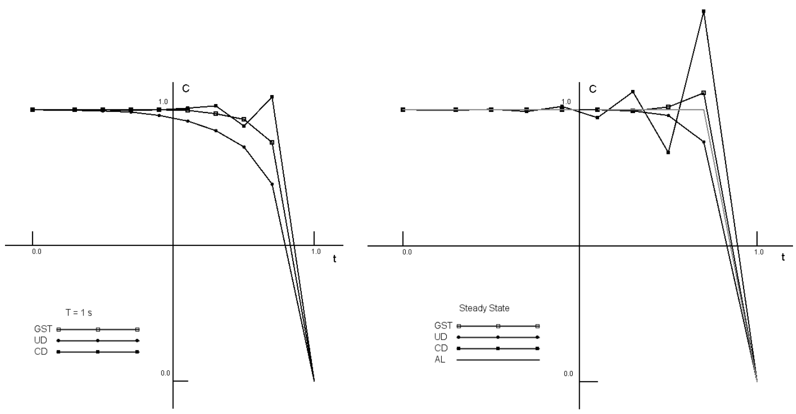

Figure 8: Analytical (AL) and numerical solutions of the advection dominant problem

As one can see the GST approximation has a good stabilisation effect in the advection dominated regime with Peclet number greater than 2 .

5.2 Two Dimensional Dam Break Induced Flow

The dam break induced flow problem is governed by the set of shallow water equations. These equations describe water flows (u,v) with a free surface (η) under the influence of gravity, thus it is a vector valued two dimensional problem. The stabilisation parameter used in this case is the one described in Eq. 20.

Following the established practice, see Bechteler et al. (1993), of testing this kind of numerical models, i.e. shallow water models, a simulation run of a dam break in two closed pools joints together with a breach, see Figure 9, is undertaken.

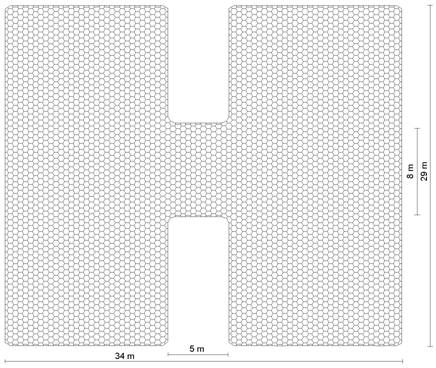

Figure 9: The computational domain with 4231 voronoi-region finite volumes

The computational domain is divided into (4231) voronoi-region finite volumes. The average size of the decomposition is $\Delta x = 0{,}5 m$. The bottom of the pools and the breach are horizontal and frictionless. The water velocity at the walls is considered equal to zero. Initially the reservoir , the left side pool, water level is 1 m high and the right side pool water level is 0.2 m high, giving a ratio of $h_l / h_r = 0{,}2$. Figure 10 shows the results of this test case at (t = 1 s) immediately after the dam break, (t = 7 s), where the bore is well-developed, and (t = 8 s), where the wave front has hit the rear side of the right pool.

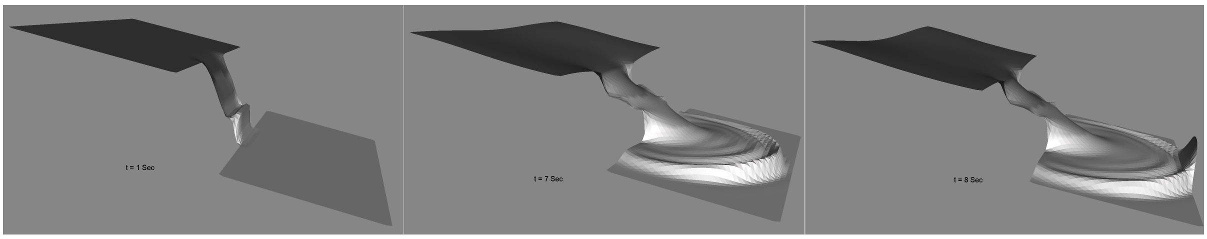

Figure 10: Two-dimensional dam-break induced flow simulation

As shown in Figure 10 the simulated dam break induced flow has a realistic propagation of the front wave without the oscillation otherwise resulting from only using the global approximation.

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