PARAMETRIC FINITE ELEMENTS
BASED ON NATURAL ELEMENT COORDINATES

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Key words: natural element coordinates, finite elements, convex cells, Delaunay decomposition, parametric cell

Abstract. *Different types of finite elements based on polygonal bounded cells as well as parametric cell are presented. A local coordinate system is introduced for such cells called natural element coordinates. The represented natural element coordinates can be used to formulate interpolation functions and test functions on convex polyhedrons in the n-dimensional Euclidian space. The interpolation on the basis of the natural element coordinates are c_0-continuous in the vertices and at least c_1-continuous within the cell. To describe warped-lined or -planed elements this interpolation functions can be used as shape functions. The presented method is applied to a standard benchmark problem. A geometrically two-dimensional square disk with a hole is subjected to a constant boundary traction acting upon two opposite sides. Additional some aspects on higher order interpolation spaces based on parametric cells are presented.*

1 INTRODUCTION

To solve a partial differential equation or a system of partial differential equations numerical methods are often used. A widely used method is the finite element method (FEM). It is based on a decomposition of the investigation area, a space of interpolation functions and a set of degrees of freedom. The classical geometrical basis of finite elements are triangles, quadrangles, tetrahedrons and hexahedrons. A generalized approach is the description of geometry using geometric cells. The use of arbitrarily dimensional convex polyhedrons as geometrical basis of finite elements increases substantially the flexibility with the generation of finite element decompositions. Sometimes this is the only way to generate a unique decomposition for a given node distribution. Convex polyhedrons in combination with a simple local coordinate system, the natural element coordinates, permits an uniform element formulation of interpolation functions and test functions on convex polyhedrons. With this polyhedrons and interpolation functions it is possible to formulate parametric finite cell elements.

2 GEOMETRICAL BASIS

The geometrical basis of the finite elements is formed by a compact non-empty set in the Euclidean space $\mathbb{R}^n$. Typical representatives of compact sets in the Euclidean space are such, which are finite and bounded. The simplest compact sets are bounded intervals in the $\mathbb{R}^1$, triangles and squares in the $\mathbb{R}^2$ as well as tetrahedron or hexahedron in the three-dimensional space. Convex polyhedrons are generalizations of the described geometrical elements.

2.1 Convex polyhedrons

Convex polyhedrons Z in the n-dimensional Euclidean space $\mathbb{R}^n$ can be described in different equivalent kinds. The definition of a convex polyhedron over the convex hull of given reference points, later on called vertices $E = \{e^1, e^2 \ldots e^N\}$, frequently forms the basis for suitable construction methods and/or algorithms. In the context of the formulation of finite elements the description of convex polyhedrons over the Minkowsky product

$$Z := \{p : p = \lambda_1 e^1 + \lambda_2 e^2 + \ldots + \lambda_N e^N, \lambda_i \geqslant 0 \wedge \sum_i \lambda_i = 1\} \tag{1}$$

appears more suitable.

The largest number of m, for which there are m linear independent points in the set of the vertices E, is called the dimension of the convex polyhedron Z. A m-dimensional convex polyhedron has a finite number of sides and each side is again a convex polyhedron. The $(m\text{-}1)$-dimensional sides of Z are called facets, the one-dimensional sides edges and the zero-dimensional sides are the vertices.

The class of the considered geometry can be extended to special non-convex polyhedrons [7], if these can be constructed over regularized set operations [1] from the convex hull of the vertices and convex sub-polyhedrons. These convex sub-polyhedrons describe the points of the convex hull, which are not contained in the non-convex polyhedron.
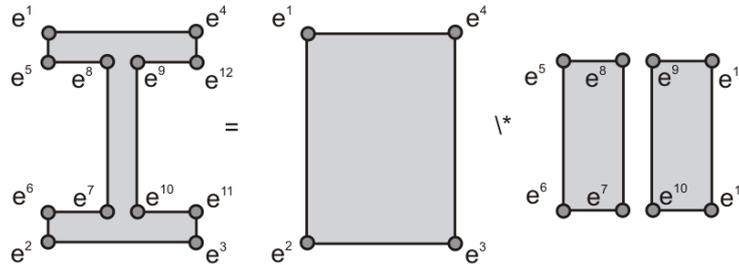
Figure 1: Construction of non-convex polyhedron

The generalized treatment of convex and non-convex polyhedrons as geometrical basis of finite elements is substantial for an uniform description of all points of a finite element. This is achieved by an uniform formulation of element coordinates.
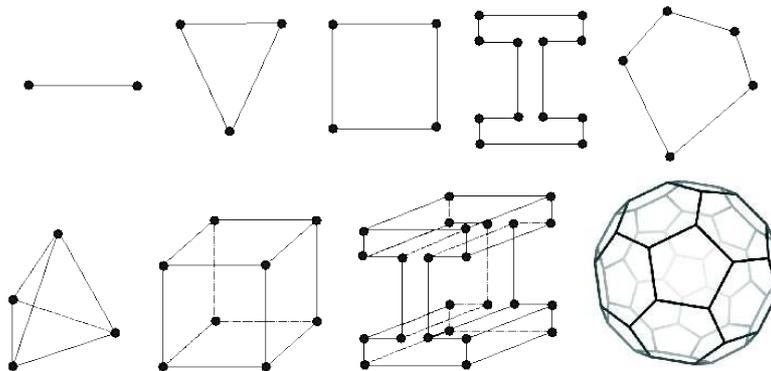

Figure 2: Polyhedrons in different dimensions

2.2   Natural element coordinates

The formulation of a local coordinate system permits an uniform element formulation in the method of the finite elements. The description of the convex polyhedron Z by the Minkowsky product (1) of its vertices E suggests to use the factors of the linear combination as element coordinates. If a m-dimensional convex polyhedron has $m+1$ linear independent vertices, the factors are unique and called barycentric coordinates. If a convex polyhedron consists of more than $m+1$ vertices, the factors are not unique. If the natural neighborhood coordinates introduced by Sibson [8] are restricted to the convex polyhedron, one receives unique natural element coordinates, which are related to the vertices of the convex polyhedron.

The determination of the natural element coordinates of a point x concerning the polyhedron Z is based on the computation of the Voronoi diagram of second order concerning the corners and the point x.
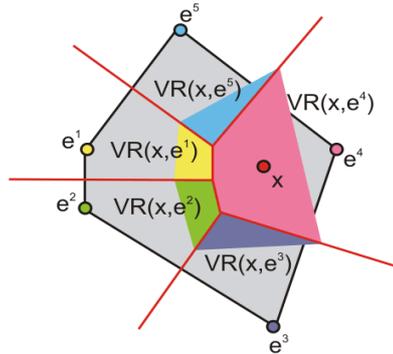
Figure 3: Voronoi decomposition of the convex
polyhedron with sub-regions

Firstly, the Voronoi decomposition of first order of a convex polyhedron is determined concerning its vertices e^i. Each vertex of the convex polyhedron has its own Voronoi region. The Voronoi region of a vertex e^i is the set of all points p which has a smaller or equal distance to the vertex e^i as their distance to the remaining vertices e^j

$$VR(e^i):=\left\{p\in\mathbb{R}^n:d(p,e^i)\leq d(p,e^j)\,\forall\,j\neq i\right\}.\tag{2}$$

The Voronoi region of second order of a convex polyhedron is determined concerning its vertices e^i and a point x of the convex polyhedron. A Voronoi region of second order is the set of points p, whose distance to the point x is smaller or equal their distance to a vertex e^i, if its distance to this vertex is smaller or equal their distance to the remaining vertices e^j

$$VR(x,e^i):=\left\{p\in\mathbb{R}^n:d(p,x)\leq d(p,e^i)\leq d(p,e^j)\,\forall\,j\neq i\right\}.\tag{3}$$

The natural element coordinates of the point x concerning the vertex e^i are determined over the Voronoi regions of second order (see Figure 3). Each Voronoi region of first or second order assigns itself a Lebesgue measure $\mu\left(VR(e^i)\right)$ or $\mu\left(VR(x,e^i)\right)$ (see [4]). This measure corresponds to the common surface area in the 2-dimensional Euclidean space. The relationship of the measure of the Voronoi region of second order of a vertex and the point x to measure of the Voronoi regions of first order of the point x concerning all vertices of the convex polyhedron is called the unique natural element coordinates

$$\lambda_i(x,e_i):=\frac{\mu\left(VR(x,e^i)\right)}{\mu\left(VR(x)\right)}.\tag{4}$$

If the considered point x lies outside of the polyhedron, then no representation in natural element coordinates exists. If the point x is accurately on a facet of the polyhedron (see Figure 4), the resulting Voronoi regions of second order have infinite measures. It can be shown by analyzing of limiting value [4] that the calculation of the natural element coordinates depends only on vertices of the facet and thus the calculation is limited to the convex polyhedron of the facet.
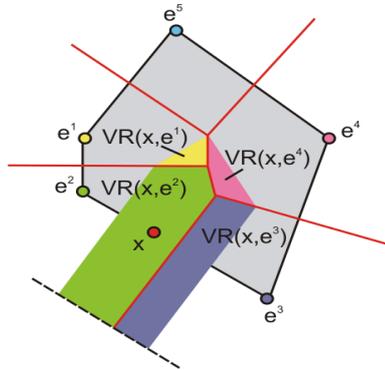
Figure 4: Calculation of the natural element coordinate on the facet

Due to the construction of the natural element coordinates of a convex polyhedron by using the Voronoi regions of second order the natural element coordinates of a point x do not inevitably depend on each vertex of the polyhedron (see Figure 5). If the Voronoi region of second order of the point x and a vertex is empty, then we say that the point x and the corner are not neighboring. The coordinate of the point x concerning this vertex has the value zero.
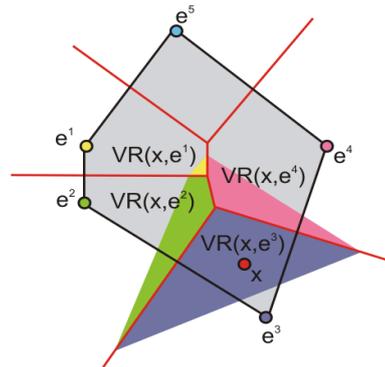
Figure 5: Influence of the coordinates

The natural element coordinates are equivalent to the barycentric coordinates of the simplex and have a bilinear characteristic inside the squares. The natural element coordinates contain the well-known local coordinate systems of classical finite element theory.

Firstly, the construction of the natural element coordinates is limited to convex polyhedrons. During the expansion to non-convex polyhedrons the occurring neighborhood relations are modified [7].
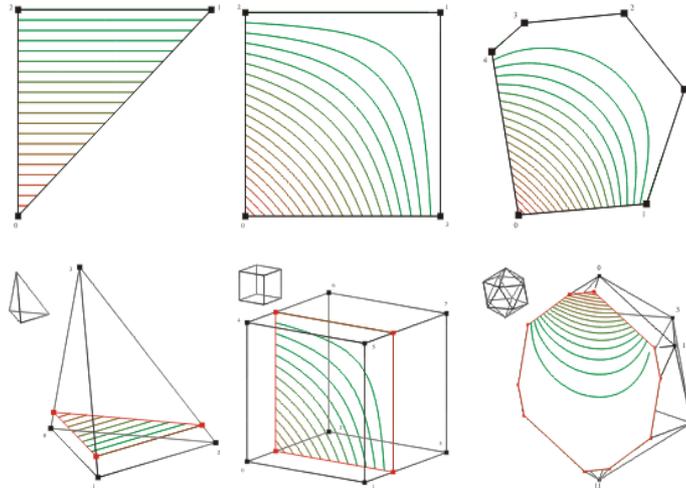
Figure 6: Characteristic of the natural element coordinates in 2- and 3-dimensional convex polyhedrons

2.3 Parametric cells

The geometry of polyhedral elements is planar bounded. Investigation areas which are describable as polyhedrons can be decomposed accurately. In order to be able to describe investigation areas with curved boundaries parametric cells as geometrical supports of finite elements are used.

The parametric cells used here are special topological cells [4]. Parametric cells are described by convex or non-convex polyhedrons and a map F (a homeomorphism). The map is described by form functions for the facets (corners, edges, sides). Its formulation in the natural element coordinates of the polyhedron is independent of the element geometry

$$x = \sum_{vertexes} N_j(\lambda(r)) + \sum_{edges} N_{ij}(\lambda(r)) + \sum_{sides} N_{ij...k}(\lambda(r)) \, . \tag{5}$$

For the classical parametric elements the map is defined always on a unit standard element. Contrary to this case, the considered parametric cells do not transform the vertices.
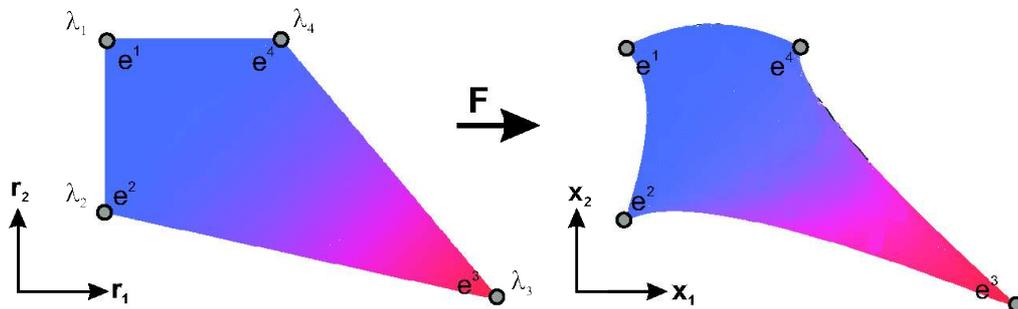

Figure 7: Parametric cell

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Thus, the map only transforms the facets of the polyhedron and has the following form:

$$x = \sum_{edges} N_{ij}(\lambda(r)) + \sum_{sides} N_{ij\ldots k}(\lambda(r)) + \ldots \quad .$$ (6)

Functions of the following kind are used as form functions on the edges:

$$N_{ij} := \left(\lambda_i(r) + \lambda_j(r)\right)^l \cdot f_{ij}(s(r))$$

with

$$s_m := \frac{\lambda_m(r)}{\lambda_i(r) + \lambda_j(r)}, \quad m \in (i,j) \quad .$$

The effect of these form functions decreases over the cell uniformly. The decay of the form function with the power l can be controlled inside the cell.

If an edge of the cell with the vertices e^i and e^j is regarded, the form of this edge can be described by the function $f_{ij}(s)$ as a function of the natural element coordinates of that edge. For each point on the regarded edge $\lambda_i + \lambda_j = 1$ is valid as well as $s_i = \lambda_i$ and $s_j = \lambda_j$. Therefore the form function $N_{ij}(\lambda(r))$ yields the deviation of the given edge from the linearity secant. At any other edges the form function vanishes.

As form function on the sides we use functions of the form

$$N_{ij\ldots k} := (\lambda_i(r) + \lambda_j(r) + \ldots + \lambda_k(r))^l f_{ij\ldots k}(s(r)) - (N_{ij}(\lambda(r)) + \ldots + N_{ki}(\lambda(r)))$$

with

$$s_m := \frac{\lambda_m(r)}{\lambda_i(r) + \lambda_j(r) + \ldots + \lambda_k(r)}, \quad m \in (i,j,\ldots,k) \quad .$$

We consider an even side of the cell with the vertices $e^i, e^j, \ldots, e^k$ and the edges $(e^i, e^j), \ldots, (e^i, e^k)$. The shape of the side can be described by the function $f_{i,j,\ldots,k}(s)$ as a function of the natural element coordinates of the side. For each point on the regarded side $\lambda_i + \lambda_j + \ldots + \lambda_k = 1$ is valid as well as $s_i = \lambda_i$ and $s_k = \lambda_k$. Therefore the form function $N_{ij\ldots k}(\lambda(r))$ yields the deviation of the given shape from shape spanned by the form function of the edges of this side. At any other sides and edges the form function vanishes.

3 INTERPOLATION FUNCTIONS

The basic idea of the finite element method is the approximation of the unknown solution by simple interpolation functions with still unknown parameters. The interpolation functions are defined on the geometric basis of the finite elements. The solution of the differential equation will be transferred to the solution of a system of algebraic equations for the unknown parameters.

The natural element coordinates allow a generalized formulation of the interpolation functions on polyhedrons and/or parametric cells. Now, we concentrate on the definition of edge-oriented interpolation functions.

The Lagrangian interpolation functions ϕ_i can be used as interpolation functions in the finite element method. Now, we present the the formulation in natural element coordinates. For edge-linear interpolations the interpolation functions consist of natural element coordinates of the associated vertices exclusively.

$$\phi_1(\lambda):=\lambda_1$$

$$\phi_2(\lambda):=\lambda_2$$

$$\phi_3(\lambda):=\lambda_3$$

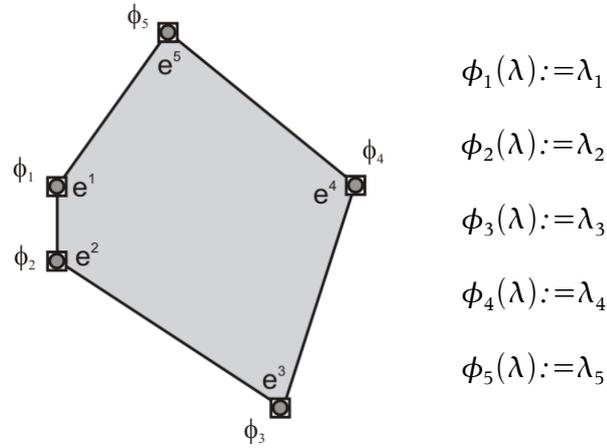$$\phi_4(\lambda):=\lambda_4$$

$$\phi_5(\lambda):=\lambda_5$$

Figure 8: Lagrangian interpolation functions - edge-linear

For edge-square interpolations the interpolation functions for the centric degree of freedom of a edge is the product of the two natural element coordinates of the associated vertices multiplied by a pre-factor. The interpolation function associated to a vertex depends on all natural element coordinates of the vertices of all outgoing edges.
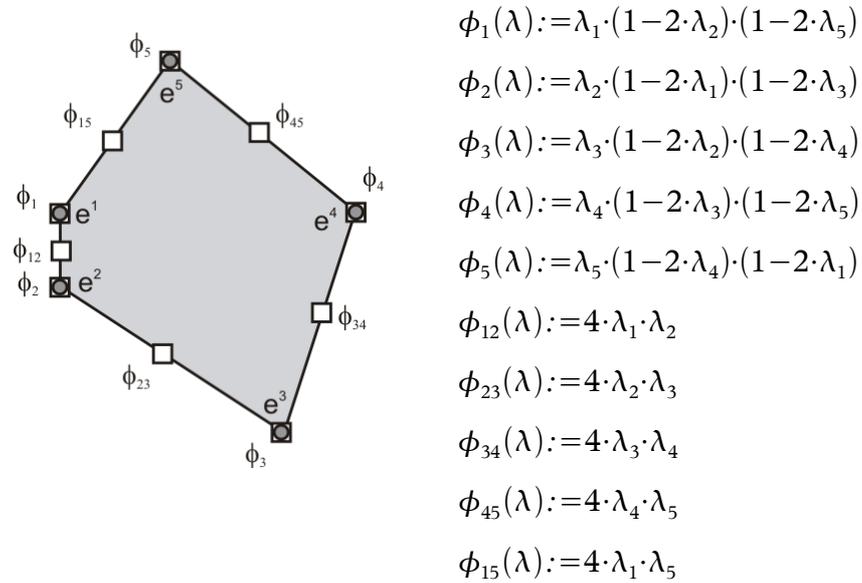
$$\phi_1(\lambda):=\lambda_1\cdot(1-2\cdot\lambda_2)\cdot(1-2\cdot\lambda_5)$$

$$\phi_2(\lambda):=\lambda_2\cdot(1-2\cdot\lambda_1)\cdot(1-2\cdot\lambda_3)$$

$$\phi_3(\lambda):=\lambda_3\cdot(1-2\cdot\lambda_2)\cdot(1-2\cdot\lambda_4)$$

$$\phi_4(\lambda):=\lambda_4\cdot(1-2\cdot\lambda_3)\cdot(1-2\cdot\lambda_5)$$

$$\phi_5(\lambda):=\lambda_5\cdot(1-2\cdot\lambda_4)\cdot(1-2\cdot\lambda_1)$$

$$\phi_{12}(\lambda):=4\cdot\lambda_1\cdot\lambda_2$$

$$\phi_{23}(\lambda):=4\cdot\lambda_2\cdot\lambda_3$$

$$\phi_{34}(\lambda):=4\cdot\lambda_3\cdot\lambda_4$$

$$\phi_{45}(\lambda):=4\cdot\lambda_4\cdot\lambda_5$$

$$\phi_{15}(\lambda):=4\cdot\lambda_1\cdot\lambda_5$$

Figure 9: Lagrangian interpolation functions - edge-square

The presented interpolation functions permit a c^0-continuity interpolation on decompositions consisting of polyhedrons and parametric cells and form the basis for finite element approximations.

4 FINITE ELEMENT APPROXIMATION

The finite element approximation is a numerical method to approximate a solution of the unknown function $u(x)$ which obeys the equation

$$F(u)=0 .$$ (11)

Now, we consider equation (11) to be an boundary value problem. Generally, equation (11) is a system of partial differential equations with associated boundary conditions.

The idea of this method is to find the unknown solution $u(x)$ not in the infinite-dimensional space but to find an optimal approximation $\hat{u}(x)$ of the solution in a finite-dimensional sub-space. The finite-dimensional sub-space is spanned by a finite set of interpolation functions ϕ_i the so called basic functions. The approximation of the unknown solution has the form

$$\hat{u}(x)=\sum_{i=1}^{N} c_i \phi_i(x) .$$ (12)

If we insert the approximation $\hat{u}(x)$ of the solution $u(x)$ into equation (11) which we like to solve, this equation is not fulfilled accurately in all cases. The occurring difference is called defect or residuum

$$\epsilon = F(\hat{u}) .$$ (13)

In order to receive an optimal approximation, the defect must be minimized. The standard Galerkin method assumes that the defect may not lie in the finite-dimensional sub-space of the approximation. The defect should be orthogonal to all basic functions ϕ_i of the interpolation space:

$$\int_{\Omega} \phi_i \epsilon = 0 .$$ (14)

The set of equations resulting from this yields the coefficients from (12).

4.1 A mechanical system

A standard benchmark problem [9] is regarded. A stretched plate with a circular central hole under plane strain conditions is considered. By using symmetry it is sufficient to discretize one quarter of the system. The measurements and boundary conditions of the system are shown in Figure 8.
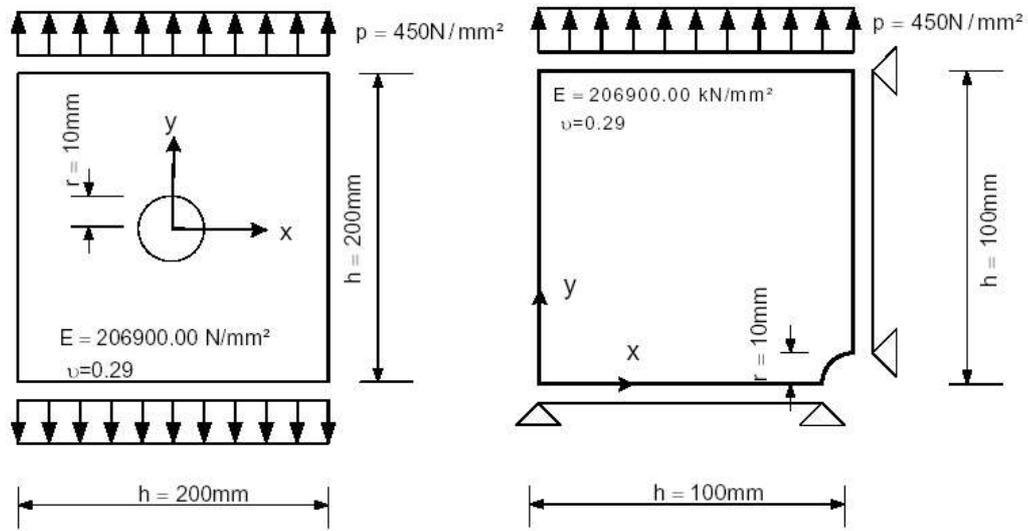
Figure 10: Whole system and one quarter system

An elastic material behavior is used as constitutive law. The strain-displacement relationships can be written as

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = L\,u \tag{15}$$

where L is the differential operator matrix and u is the displacement vector.

The relation between strains and stresses is given by the linear elastic law

$$\sigma = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = C\,\varepsilon \tag{16}$$

where

$$C = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & 0 \\ \dfrac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2(1-\nu)} \end{bmatrix} \tag{17}$$

being the elasticity matrix with Poisson's ratio ν and Young's modulus E.

The equilibrium condition is given as

$$L^T \sigma + f = 0 \tag{18}$$

where $\mathbf{f}$ is the body force vector.

Multiplying the equilibrium condition with a test function v and integrating over the domain Ω, we obtain the weak formulation

$$\int_{\Omega} [\boldsymbol{Lv}]^T \boldsymbol{C} [\boldsymbol{Lu}] d\Omega = \int_{\Omega} \boldsymbol{v}^T \boldsymbol{f} d\Omega + \int_{\Gamma_N} \boldsymbol{v}^T \bar{\boldsymbol{t}} d\Gamma \qquad (19)$$

where $\bar{\boldsymbol{t}}$ are surface tractions on Dirichlet boundary Γ_N.

In the following the displacements and the stresses for two different decompositions are illustrated. The results correspond essentially with published solutions [9].
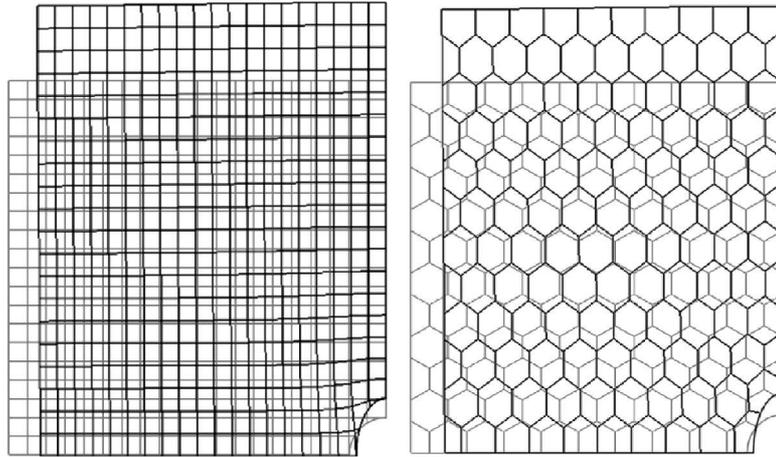

Figure 11: Grid and displacement multiplied by hundert

The calculations were accomplished with edge-square interpolation functions. The calculation grid consists of a regular rectangle grid with 1273 degrees of freedom and of a quasi-regular hexagon grid with 627 degrees of freedom. In the case of less degrees of freedom the quasi-regular hexagon grid provides a better approximation for the maximum displacement and the displacement at the edge of hole.
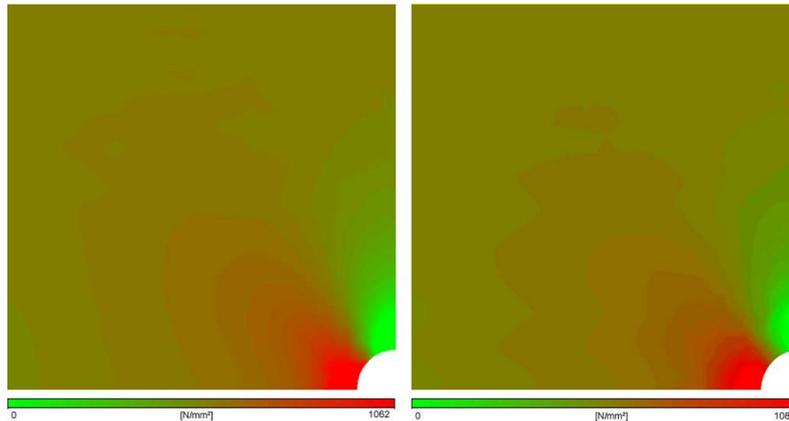

Figure 12: Contour plot of stresses

In the Figure 12 one can see the contour plot of the stress σ_{yy} and the concentration of the stress at the hole. The maximum stress appears at the edge of hole.

5 CONCLUSION

Generalized finite elements are presented. Their basis are general convex and non-convex polyhedrons as well as parametric cells. Introducing the natural element coordinates on polyhedrons and parametric cells a generalized formulation of form functions and interpolation functions could be achieved. These finite elements are suitable for the interpolation and for the approximation of solutions of partial differential equations as well. The example benchmark problem "stretched plate with a circular central hole" shows the use of the presented concept.

REFERENCES

[1] S. Abramowski, H. Müller: *Geometrisches Modellieren*, BI-Wissenschaftsverlag, 1991

[2] A. Bowyer: *Computer Dirichlet tesselations,* The Computer Journal, 24 No.2, pp. 162-166, 1981

[3] G. Farin: *Surfaces over Dirichlet tesselations,* Computer Aided Geometric Design, 7 (1-4), pp. 281-292, 1990

[4] P. Milbradt: *Algorithmische Geometrie in der Bauinformatik*, Habilitationsschrift, Universität Hannover, 2001

[5] P. Milbradt, A. Schwöppe: *Finite Element Approximation auf der Basis geometrischer Zellen,* IKM, Weimar 2003

[6] J. Schierbaum, A. Schwöppe: *Ansatzfunktionen höherer Ordnung für Finite konvexe Zellen*, Forum Bauinformatik 2003, Shaker-Verlag Aachen, 2003

[7] A. Schwöppe, P. Milbradt: *A Class of Convex, Polygonal Bounded Finite Elements;* WCCM V, Wien 2002

[8] R. Sibson: *A vector identity for the Dirichlet tesselation,* Mathematical Proceedings of the Cambridge Philosophical Society 87, (1980), pp. 151-155

[9] E. Stein: *Error-controlled Adaptive Finite Elements in Solid Mechanics*, John Wiley 6 Sons Ltd., 2002

[10] D. F. Watson: *Computing the n-dimensional Delaunay tesselation with application to Voronoi polytopes*, The Computer Journal, vol. 24, No. 2, (1981), pp. 167-172